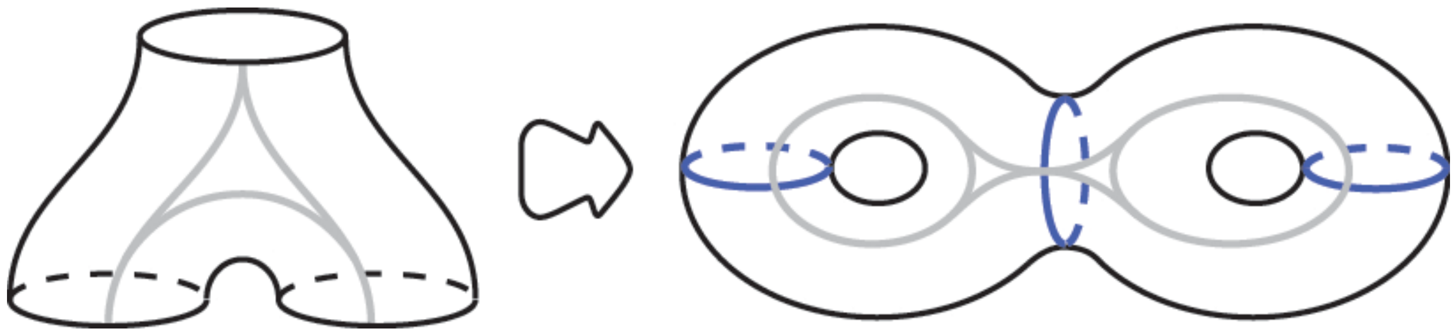


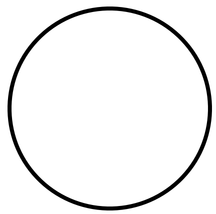
# The Topology of Trousers

Tara S. Holm
Cornell University

*MATH 5080: Mathematics For Secondary School Teachers*
*2 March 2013*

# What is .... geometry?

The study of shapes, sizes, relative positions, and the properties of space.



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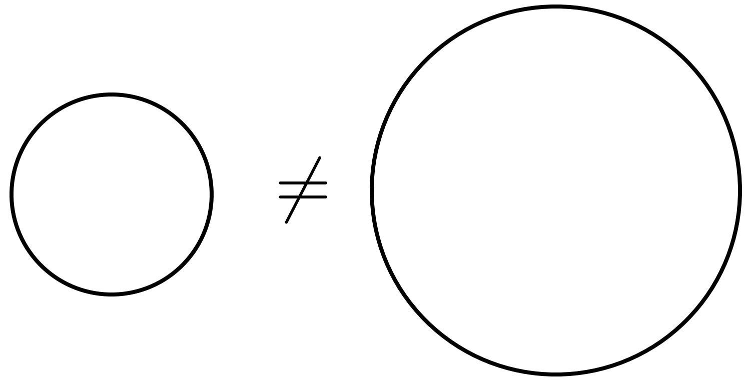

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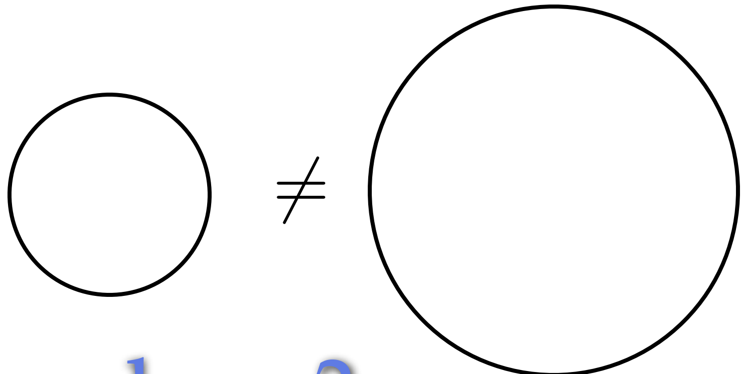

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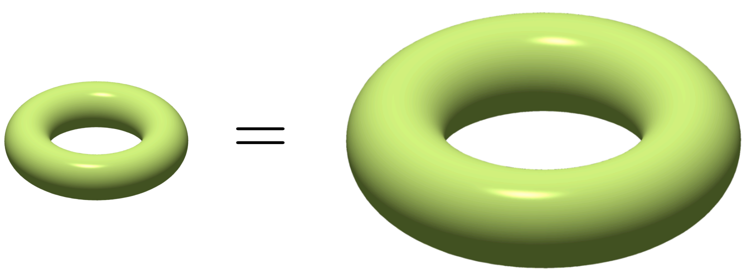

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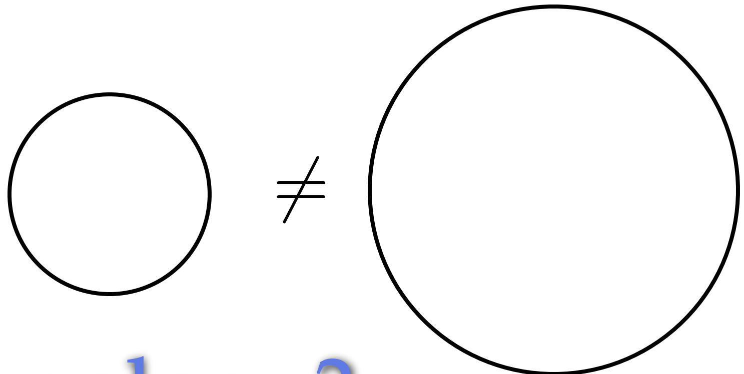

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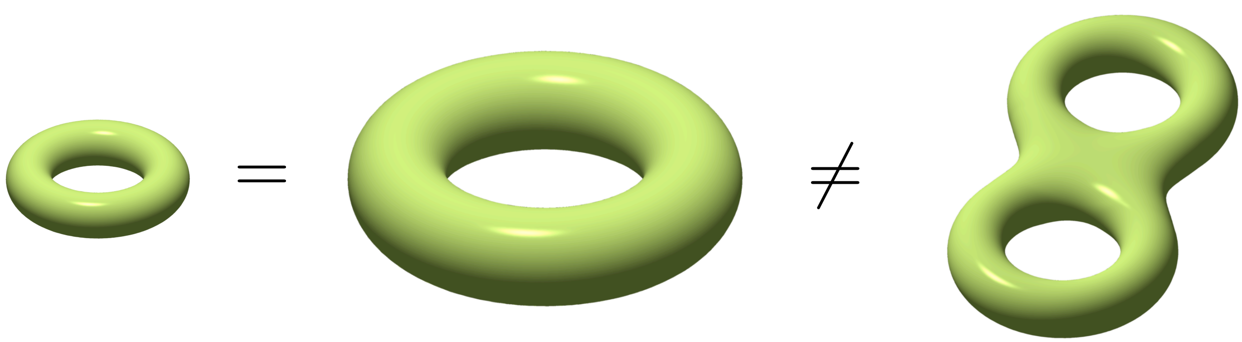

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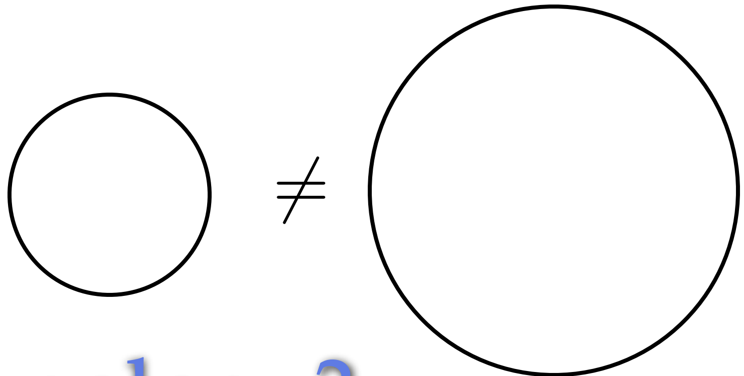

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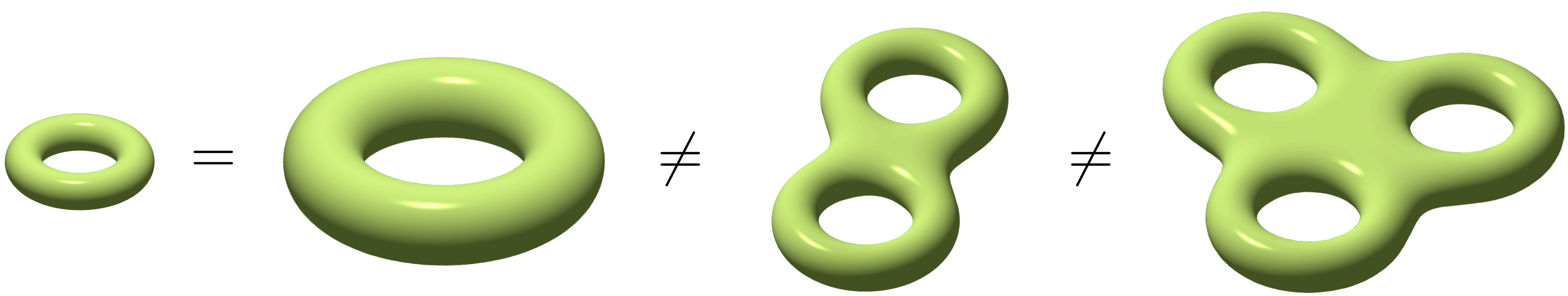
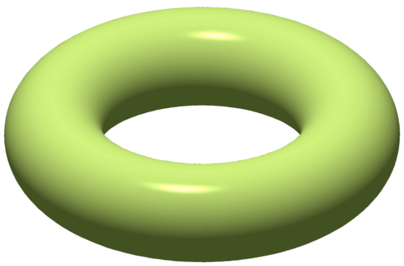
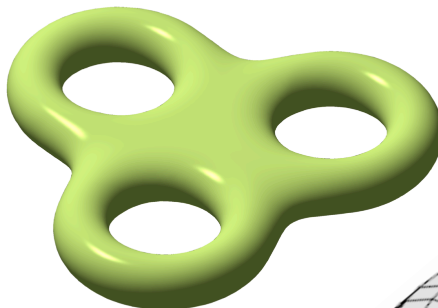
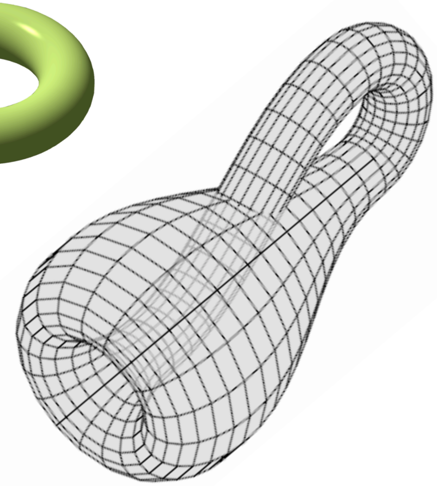
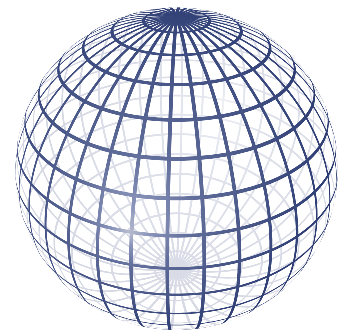
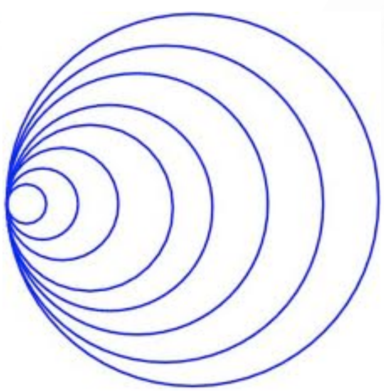
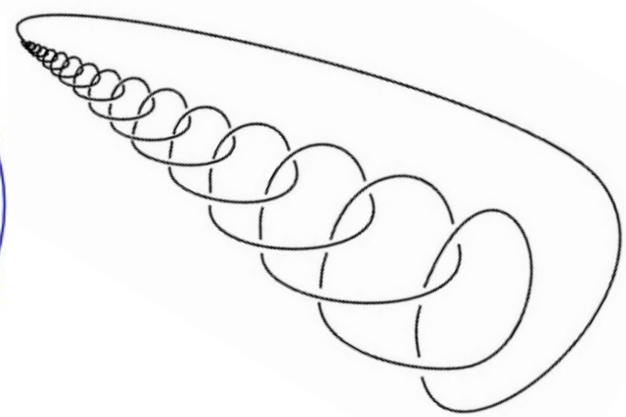
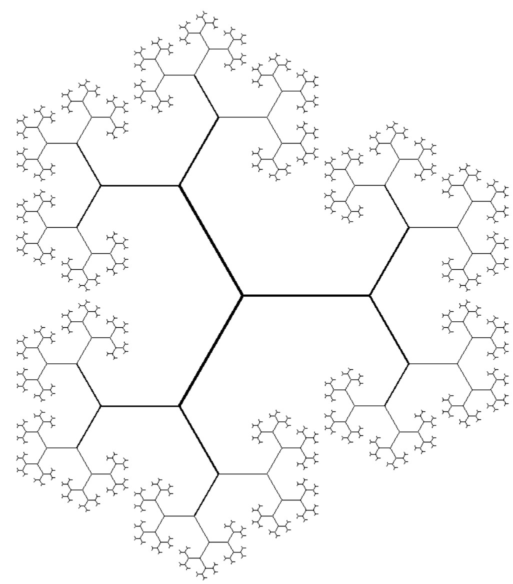


# Topological spaces

Definition: A topological space consists of a set X together with a notion of "close".



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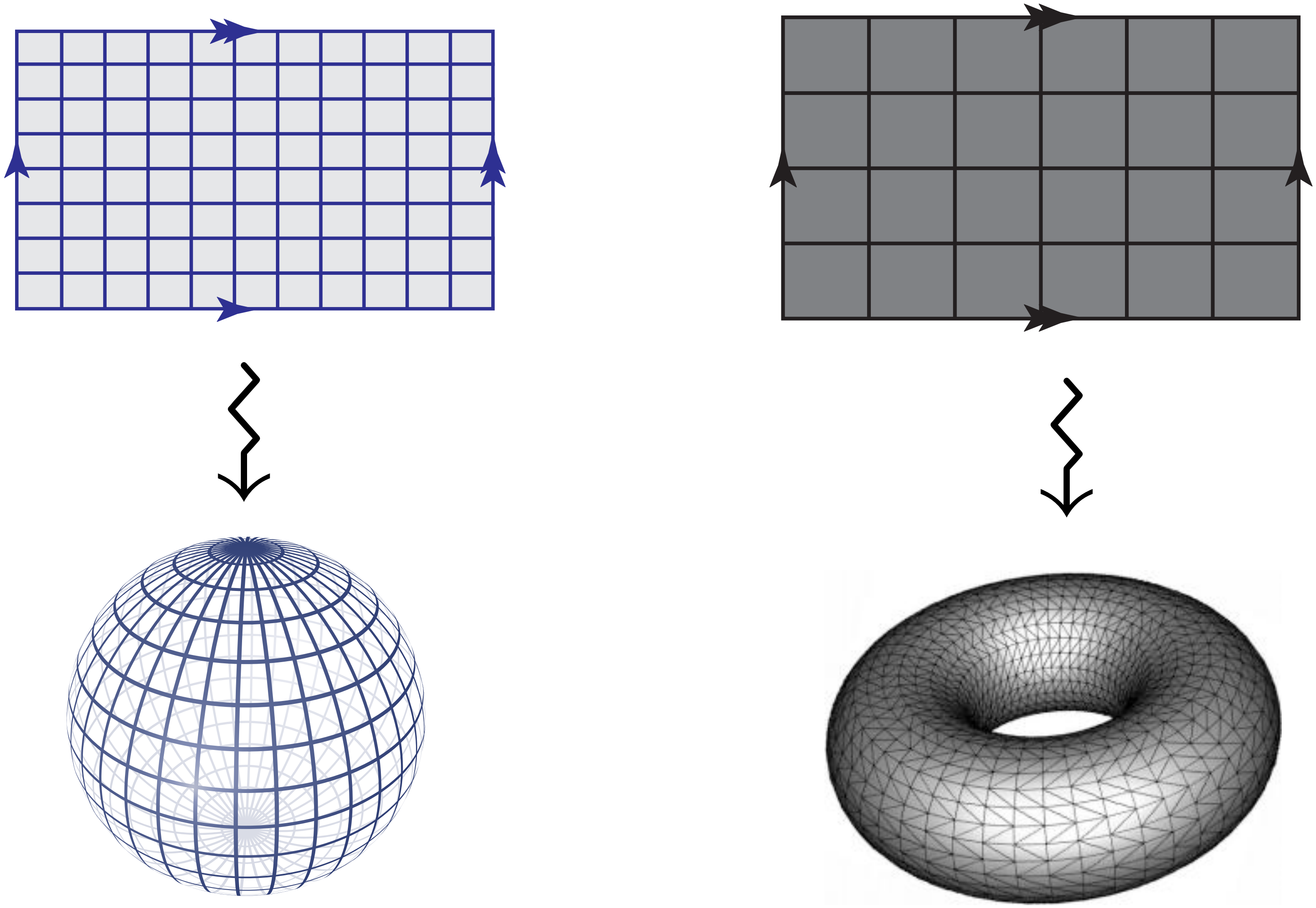

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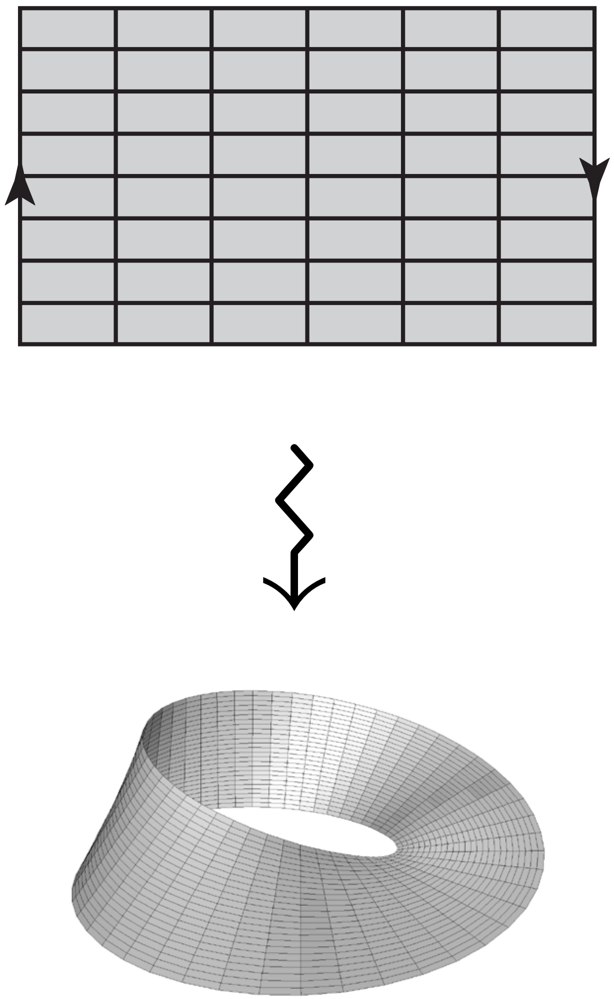

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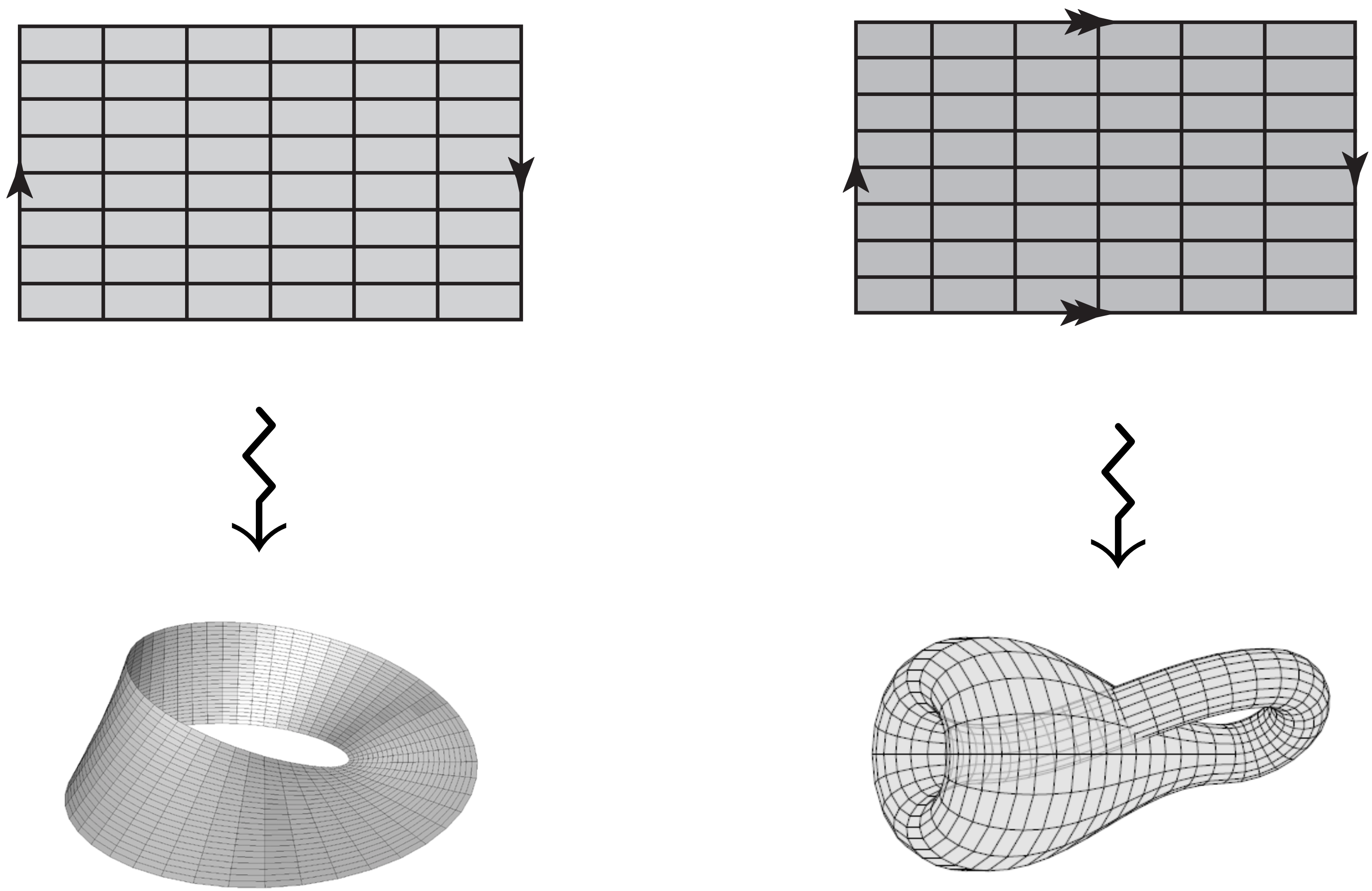

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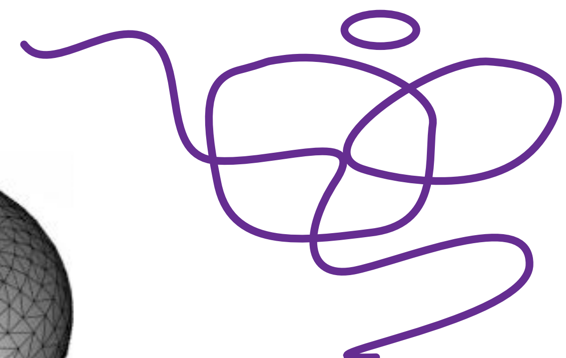

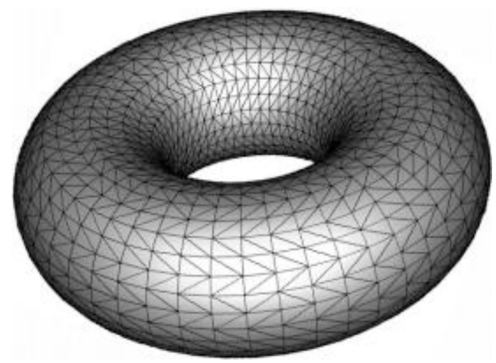

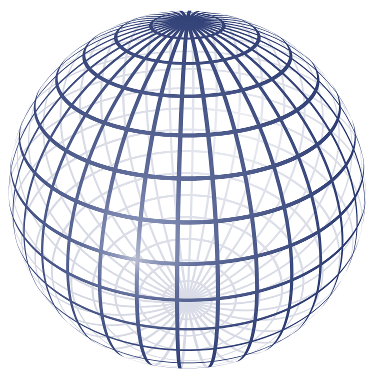

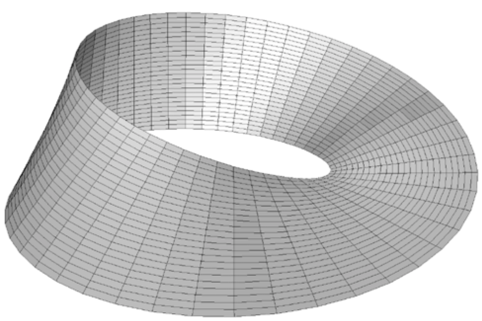

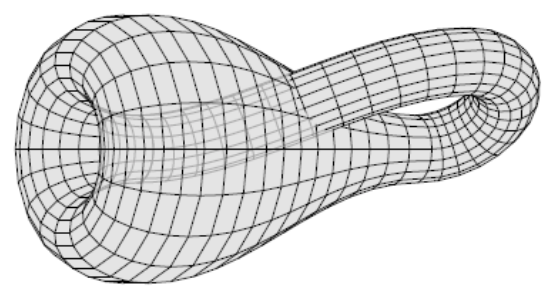

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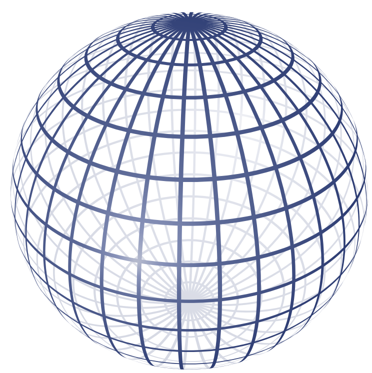
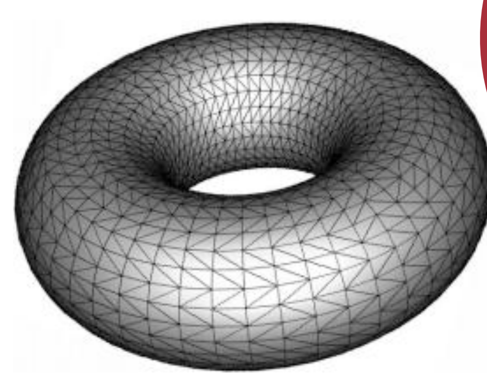

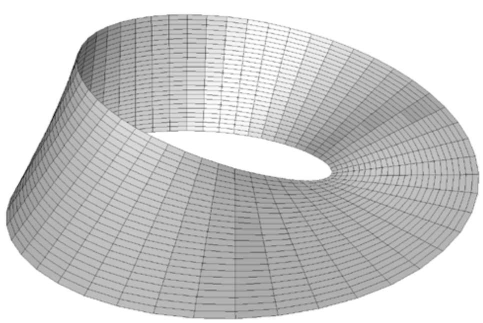
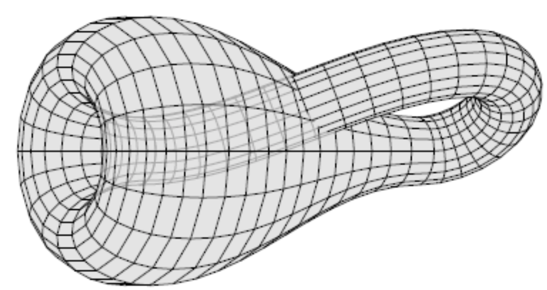

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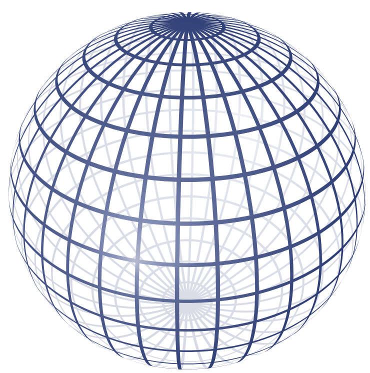

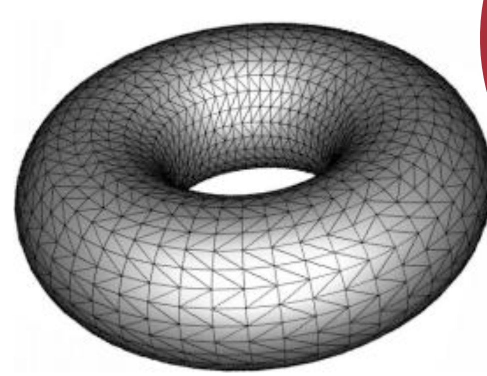

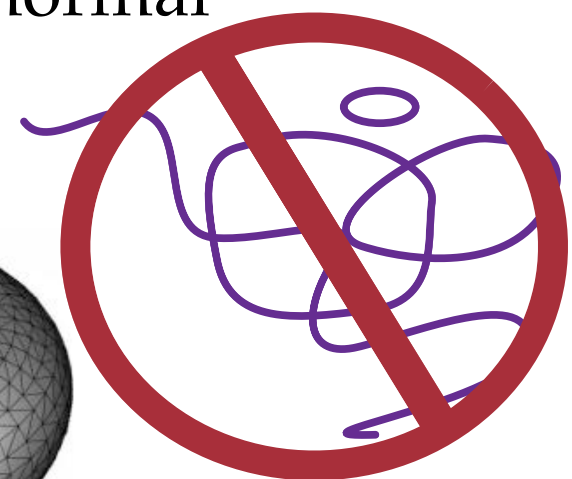

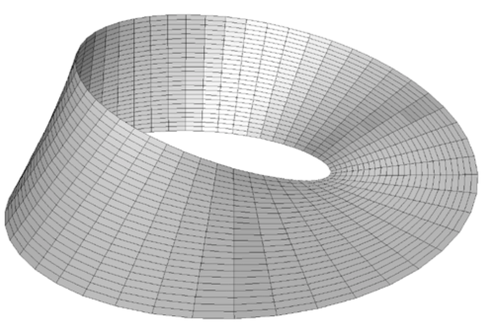

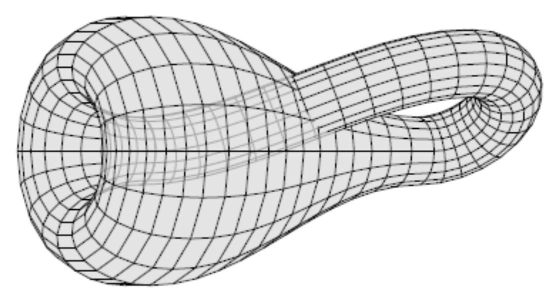

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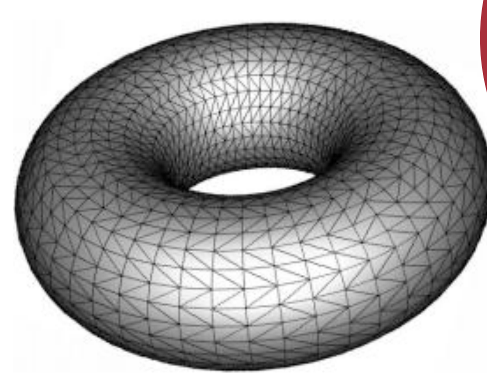

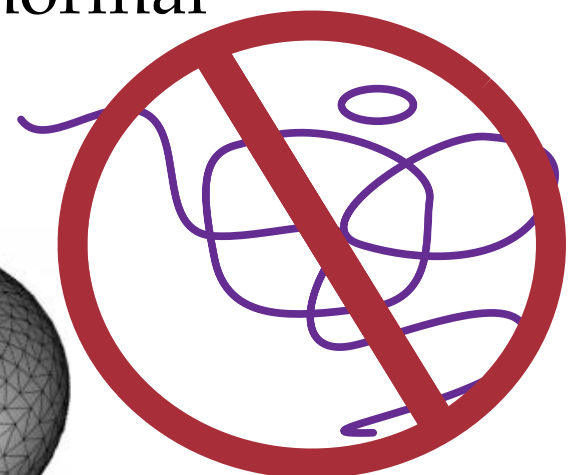

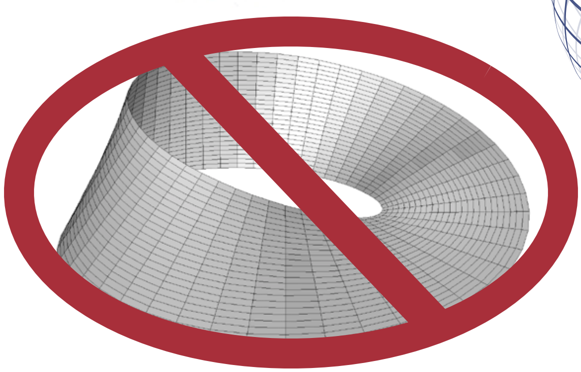

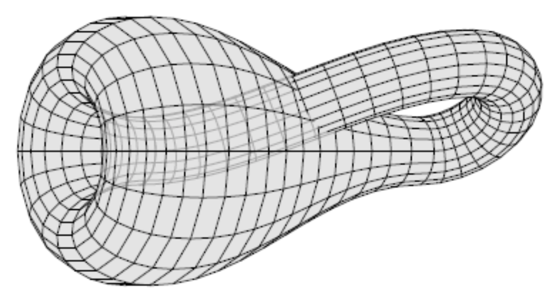

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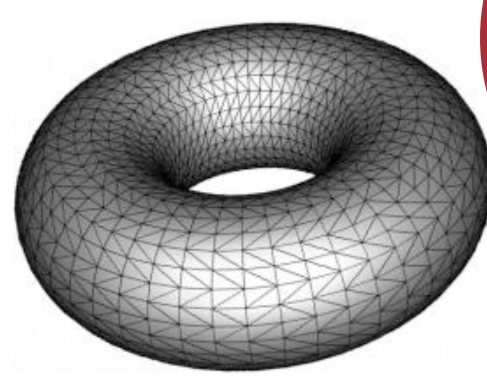

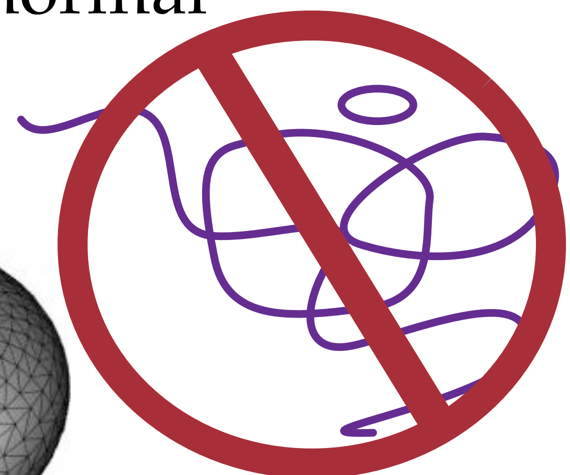

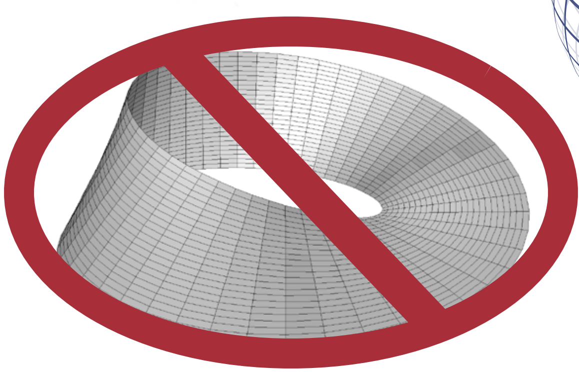

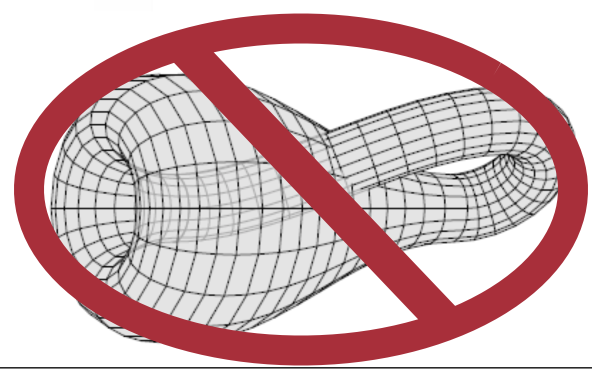

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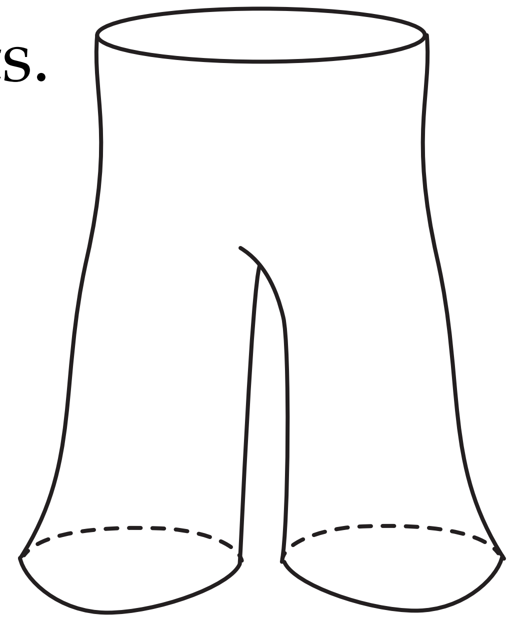

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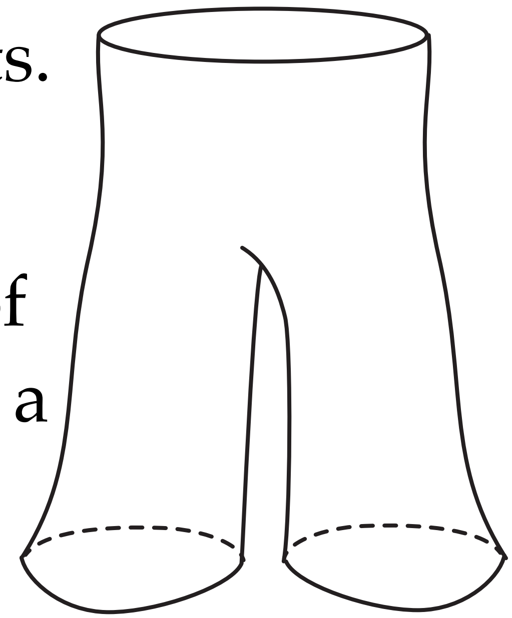

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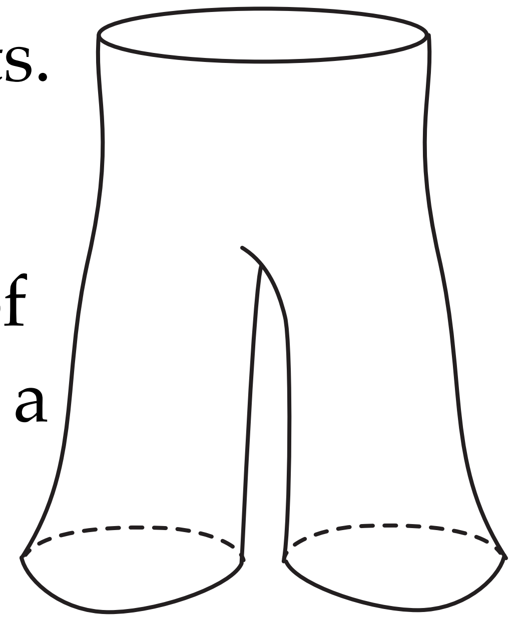

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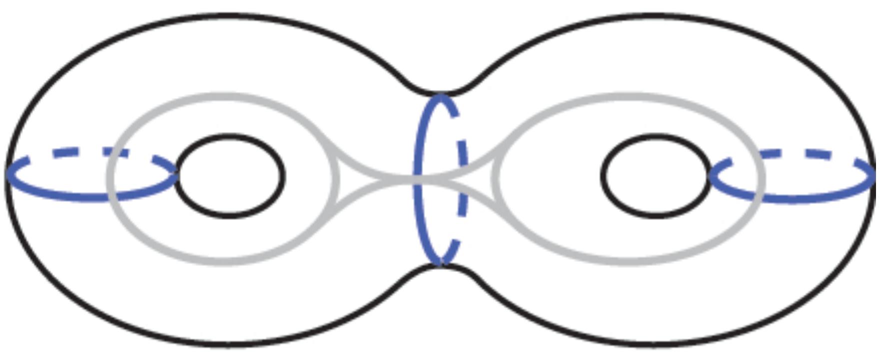

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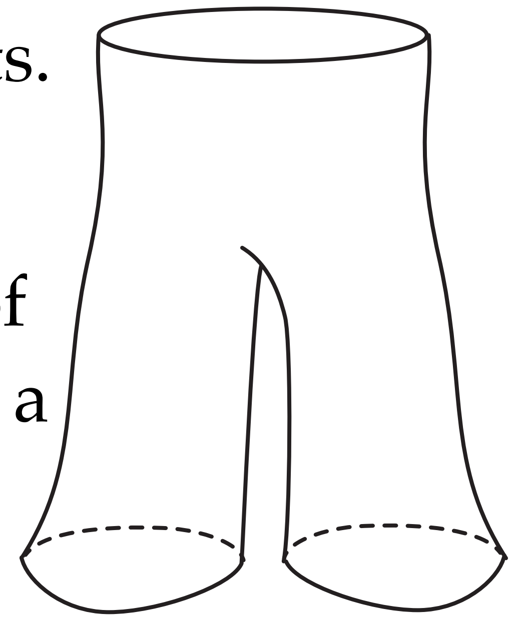

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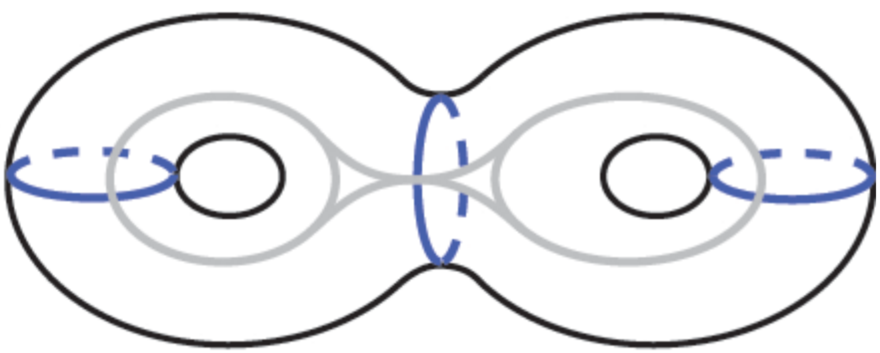

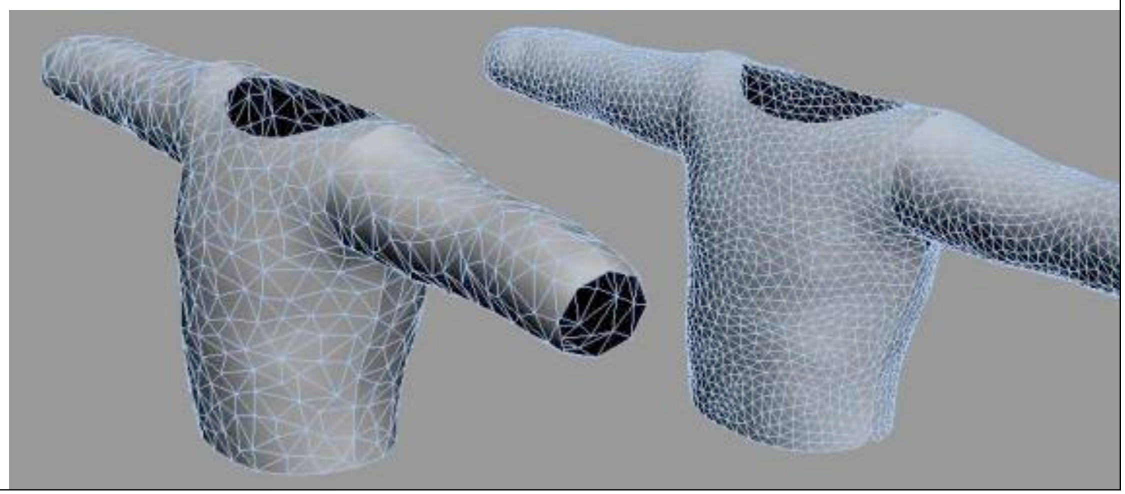

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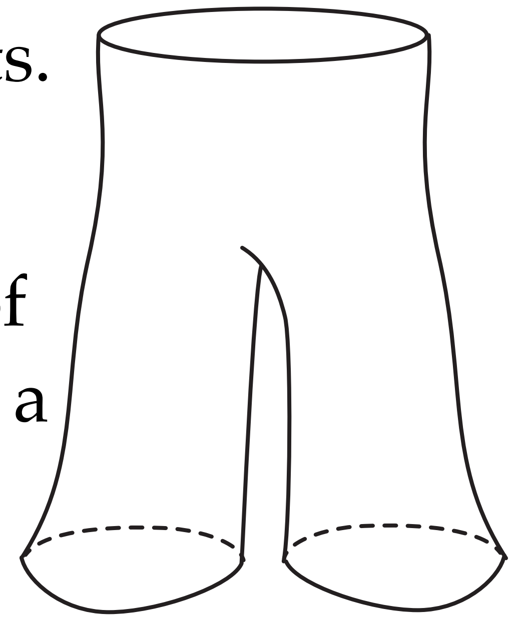

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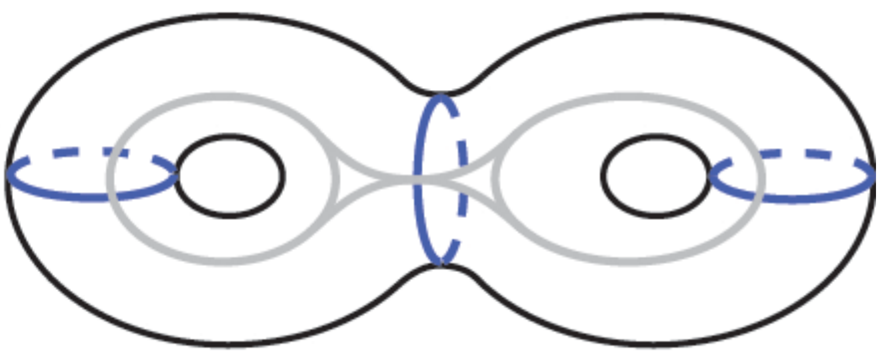

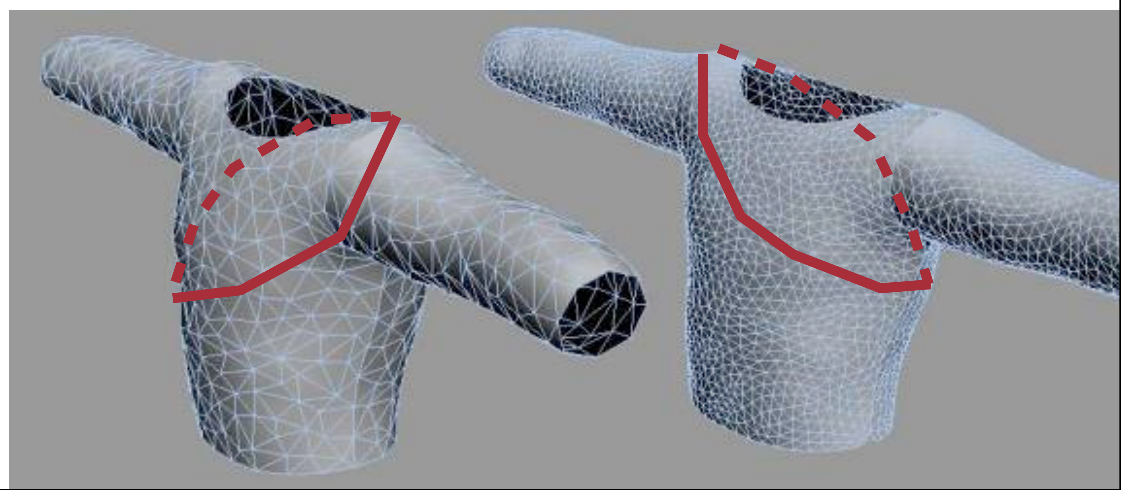

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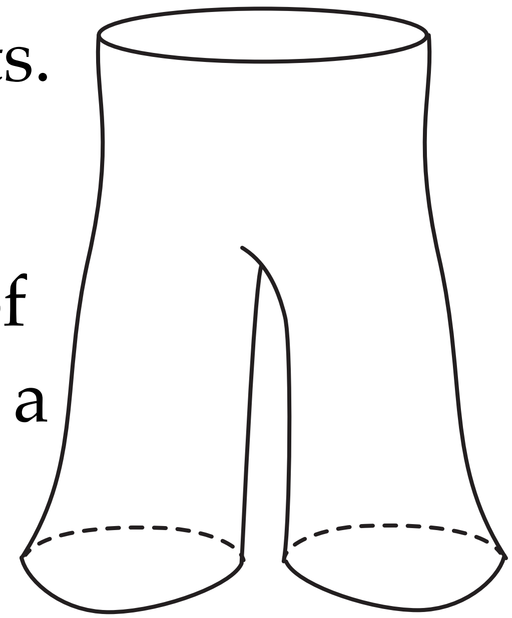

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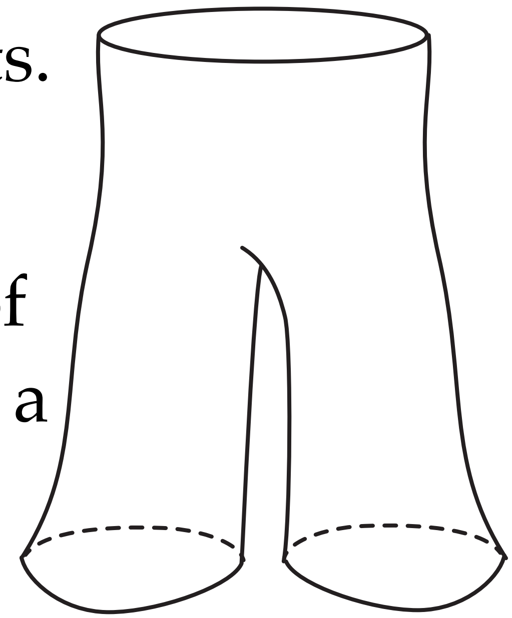

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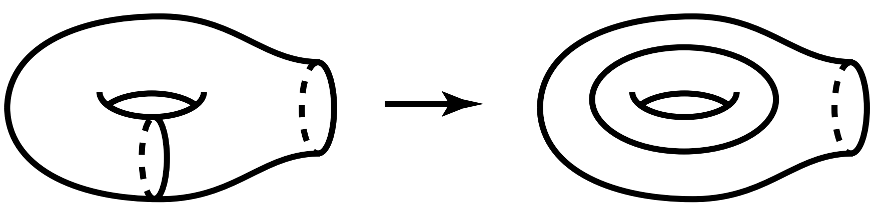
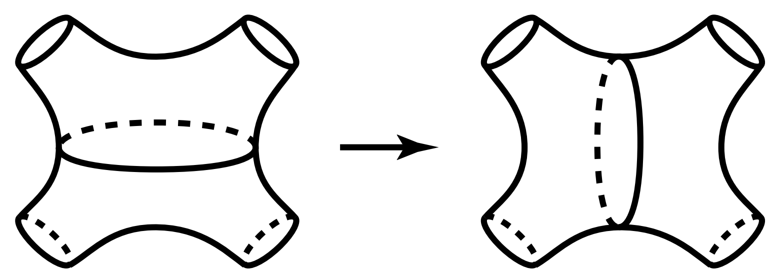


An S-move



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Definition: Two topological spaces X and Y are homeomorphic if there is a continuous bijection

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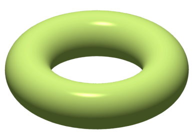

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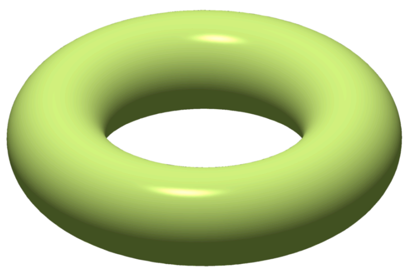

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$$\begin{array}{rcl} f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) & \to & \mathbb{R} \\ x & \mapsto & \tan(x) \end{array}$$

$$f^{-1}(x) = \tan^{-1}(x) = \arctan(x)$$

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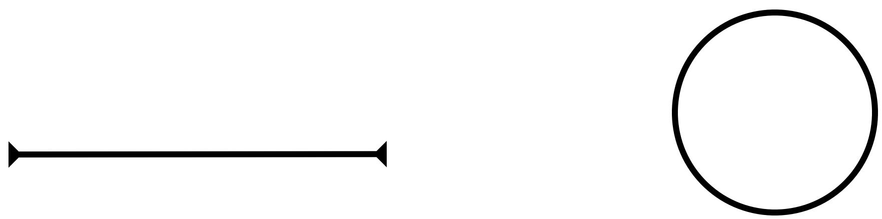

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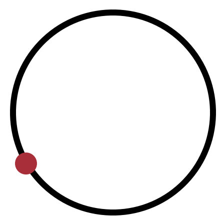

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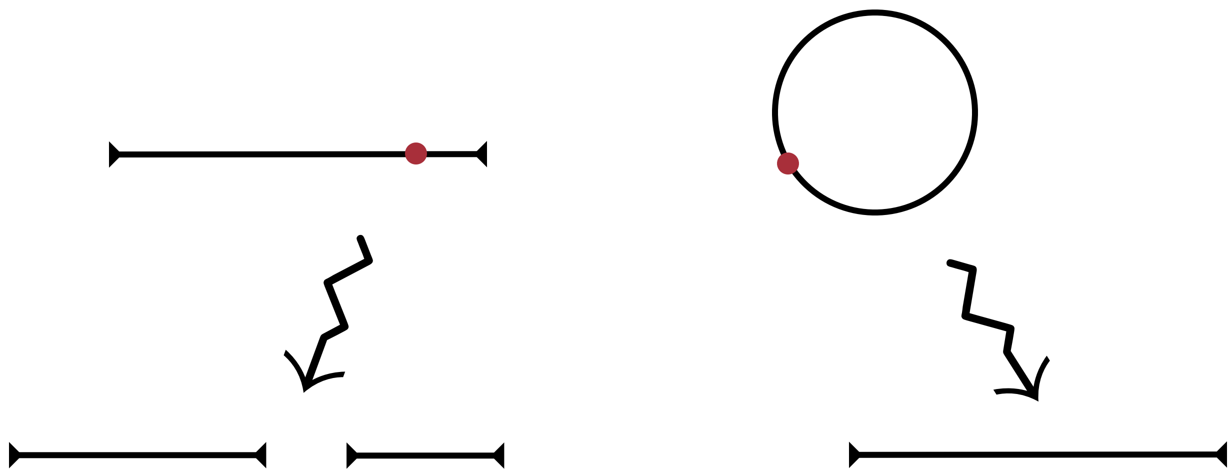

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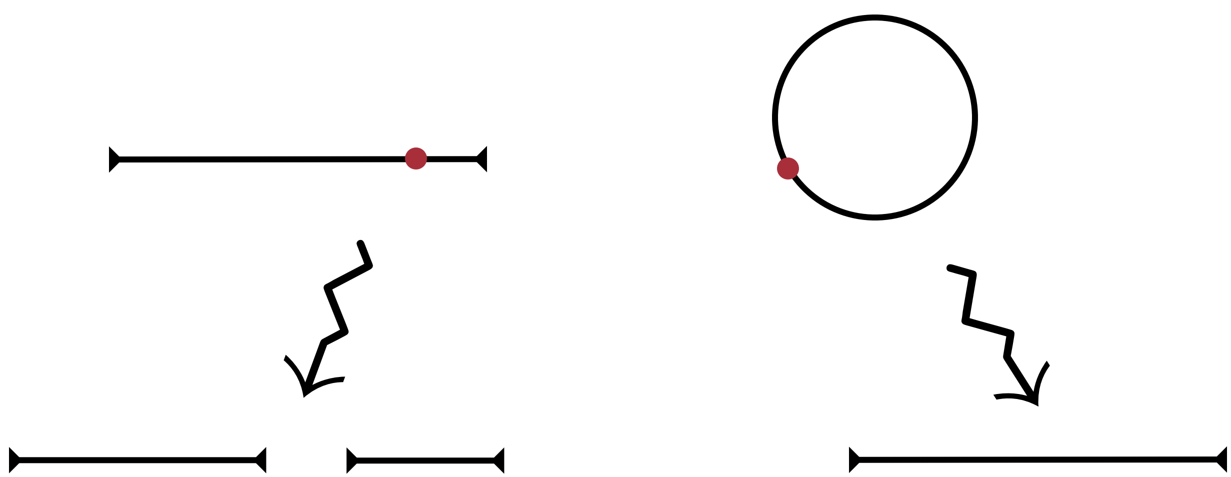

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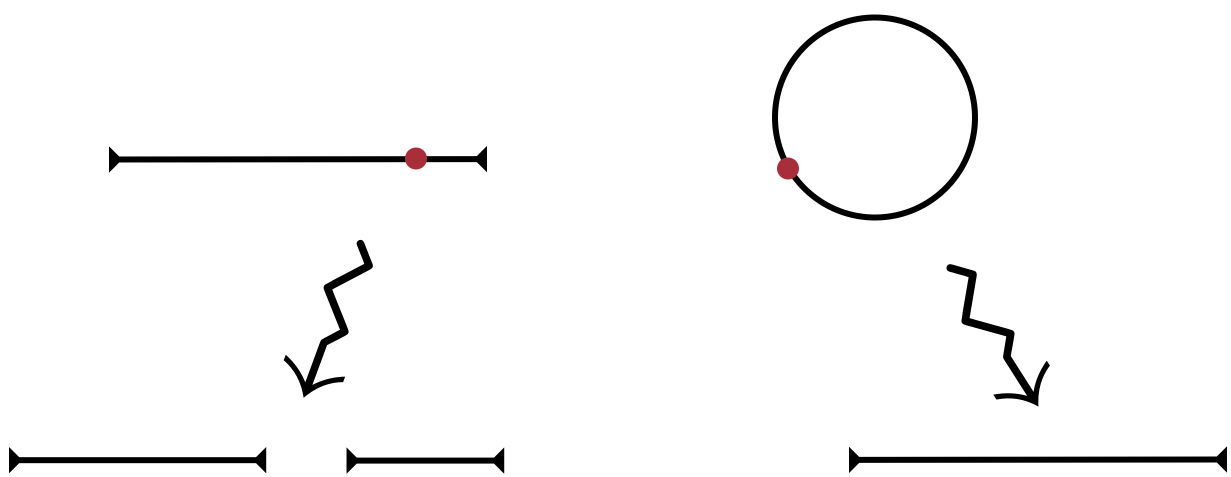

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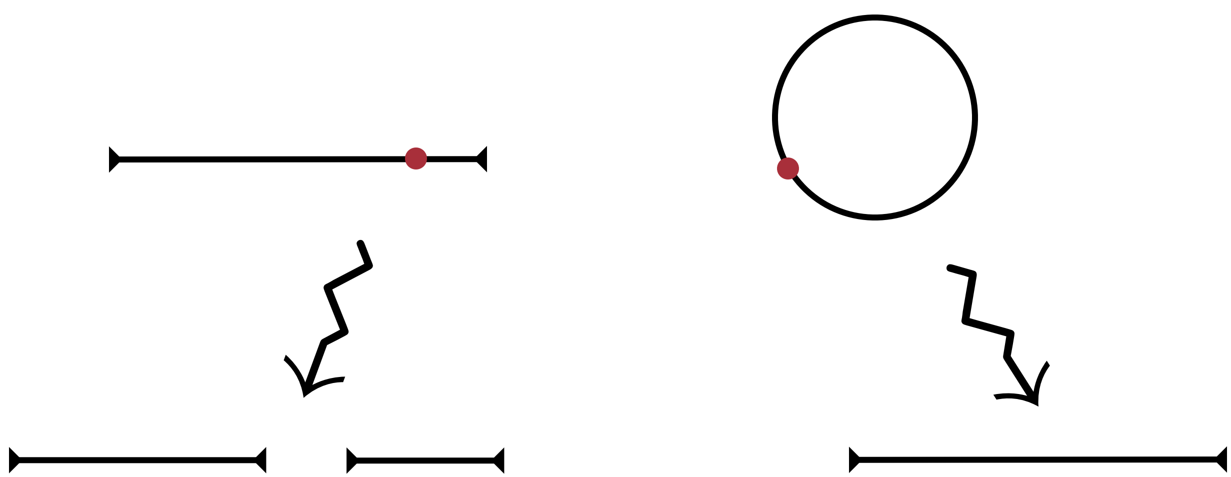

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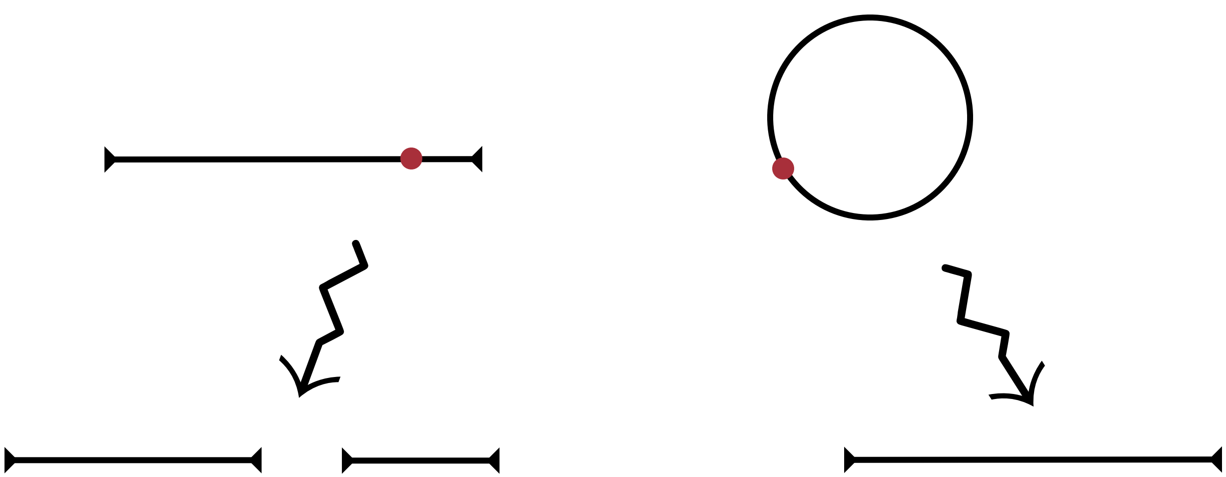

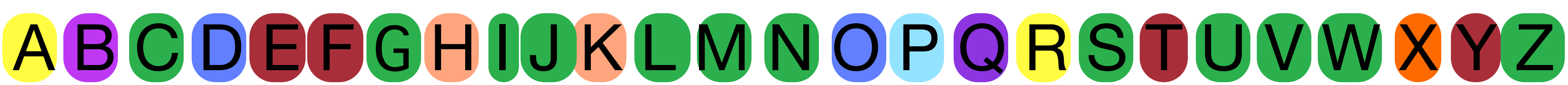

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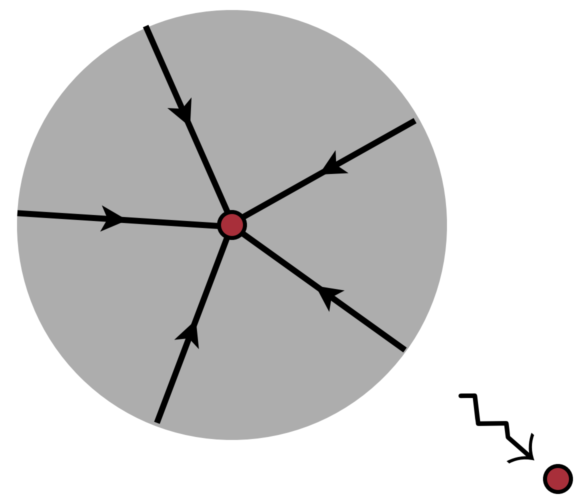

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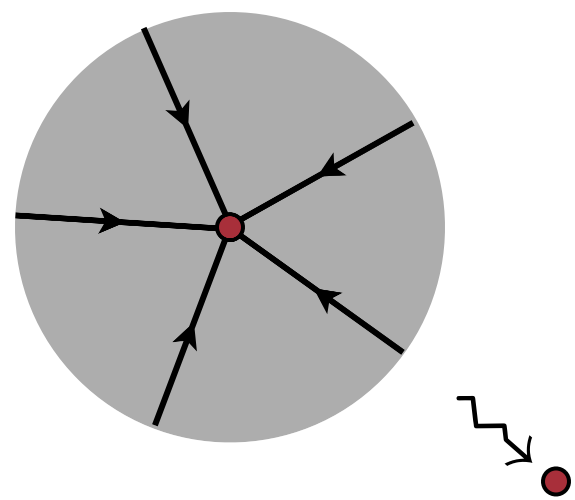

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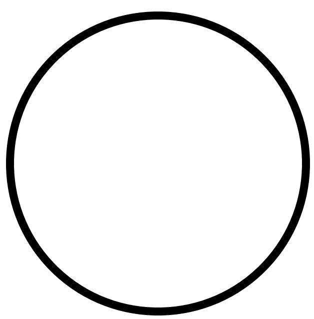

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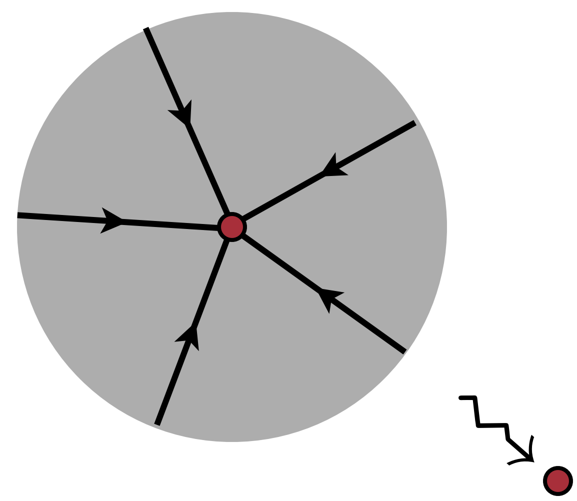

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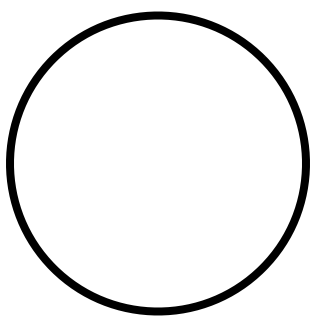

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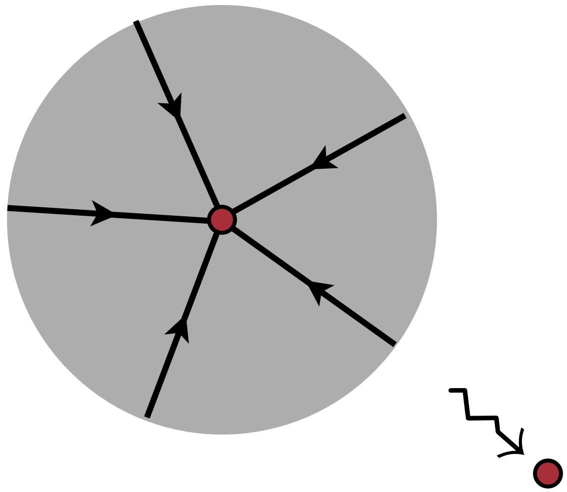

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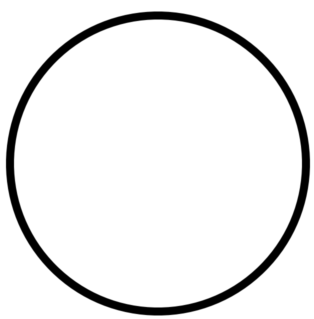

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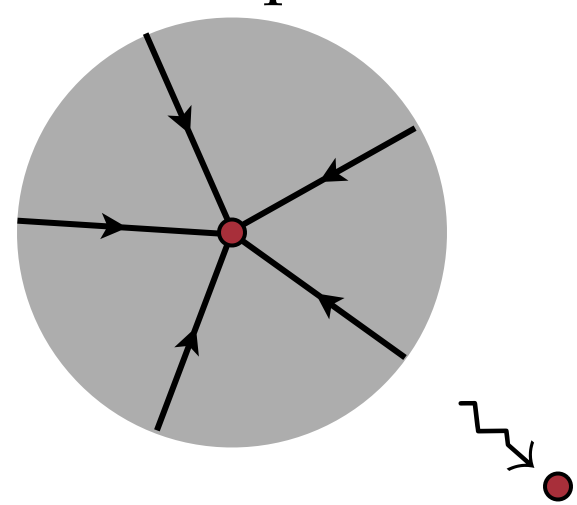

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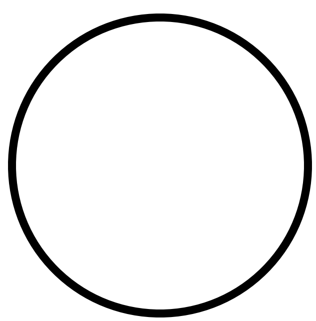

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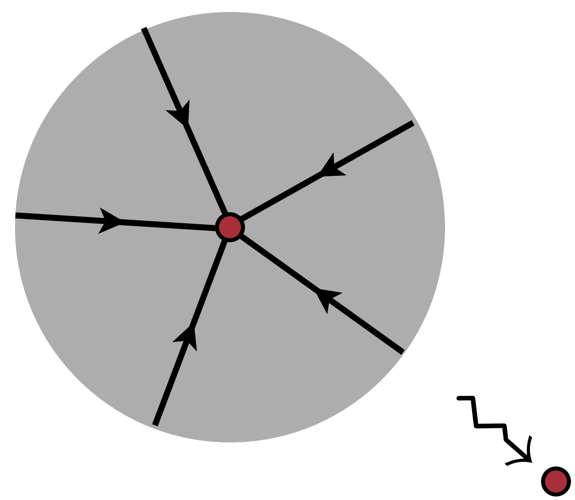

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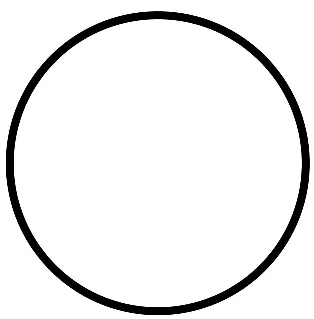

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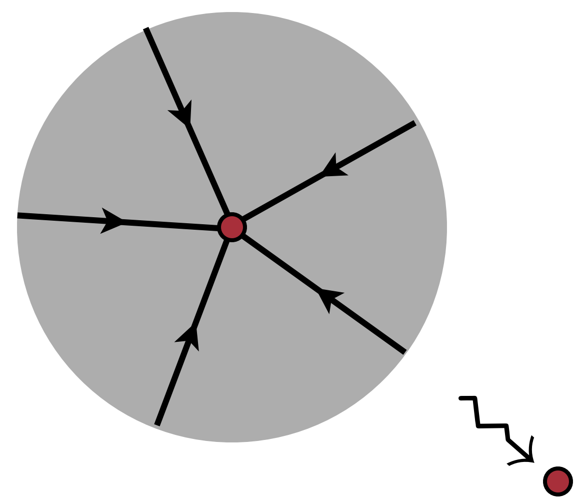

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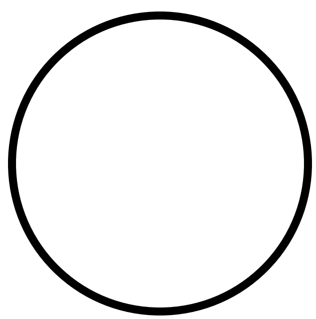

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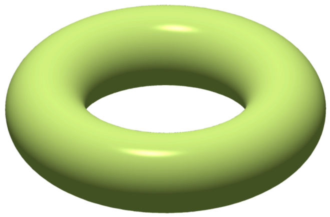

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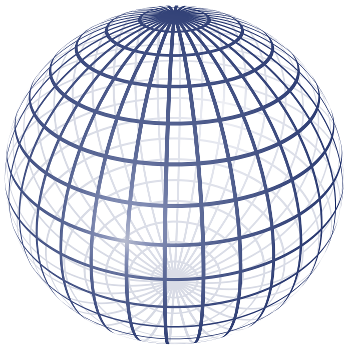

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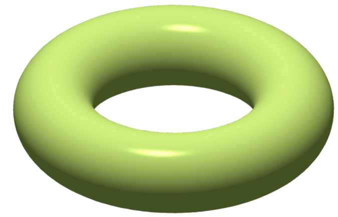

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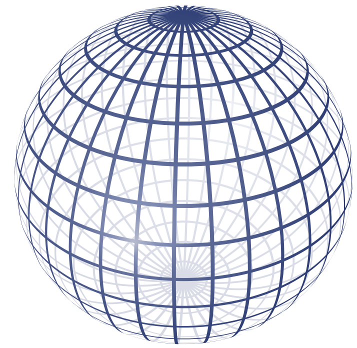

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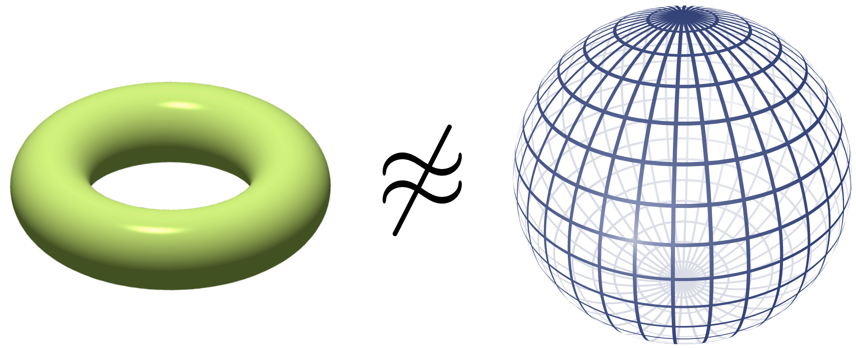

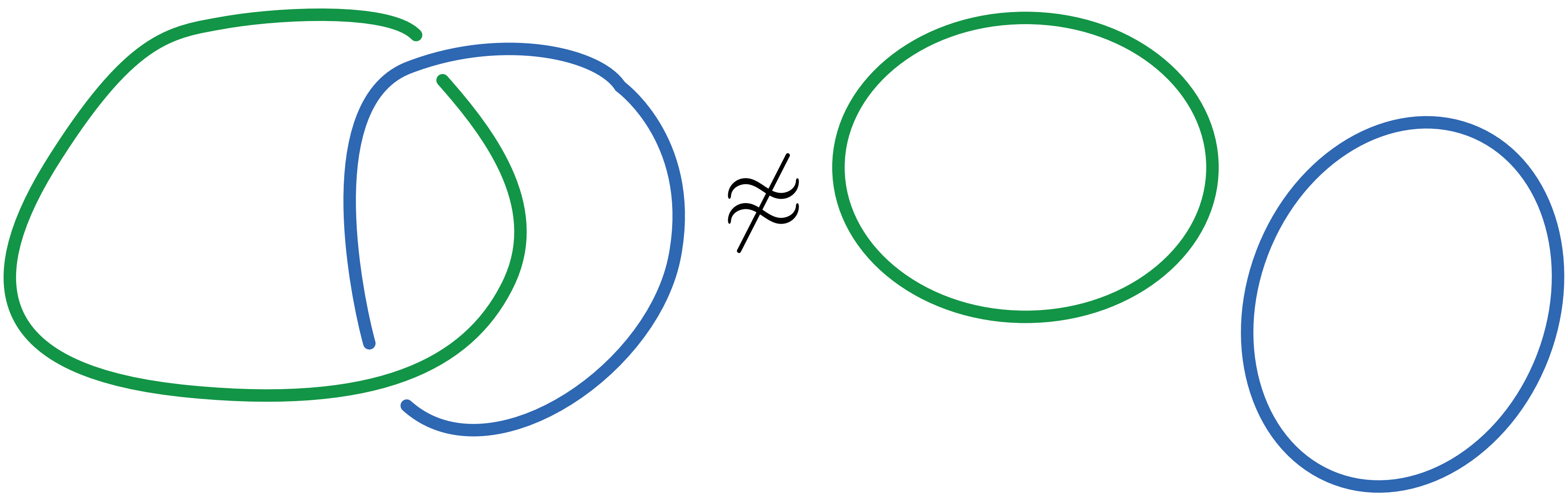

# Topologist

Definition: A topologist is a person who can turn their pants inside out without taking them off.

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**Theorem:**

Tara Holm is a topologist.

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A B C D E F G H I J K L M

N O P Q R S T U V W X Y Z

# Homotopy equivalence

A B C D E F G H I J K L M

N O P Q R S T U V W X Y Z