Mathematics 615, Fall, 2005

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Contents

1 Hilbert space and orthonormal bases

1.1 Norms, inner products and Schwarz inequality

Definition. A *norm* on a real or complex vector space, V is a real valued function, $\|\cdot\|$ on V such that
1. a. $\|x\| \geq 0$ for all x in V.
b. $\|x\| = 0$ only if $x = O$.
2. $\|\alpha x\| = |\alpha|\|x\|$ for all scalars α and all x in V.
3. $\|x + y\| \leq \|x\| + \|y\|$.

Note that the converse of 1.b follows from 2. because $\|O\| = \|0O\| = 0\|O\| = 0$.

Examples
1.1 $V = R$, $\|x\| = |x|$.
1.2 $V = \mathbb{C}$, $\|x\| = |x|$.
1.3 $V = R^n$ with $\|x\| = \sqrt{\sum_1^n x_j^2}$ when $x = (x_1, \dots, x_n)$.
1.4 $V = \mathbb{C}^n$ with $\|x\| = \sqrt{\sum_1^n |x_j|^2}$ when $x = (x_1, \dots, x_n)$.
1.5 $V = C([0,1])$. These are the continuous, complex valued functions on $[0,1]$.

Here are two norms on V.

$$\|f\|_1 = \int_0^1 |f(t)|dt \tag{1.1}$$ H5.1

$$\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\} \tag{1.2}$$ H5.2

Exercise 1.1 Prove that the expression in (1.2) is a norm.

Here is an infinite family of other norms on R^n. Let $1 \leq p < \infty$. Define

$$\|x\|_p = \Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p} \tag{1.3}$$ H5.3

FACT: These are all norms. And here is yet one more. Define

$$\|x\|_\infty = \max_{j=1,\dots,n} |x_j| \tag{1.4}$$ H5.4

Exercise 1.2 Prove that $\|x\|_1$ and $\|x\|_\infty$ are norms. (Sadly, the proof that $\|x\|_p$ is a norm is not that easy. So don't even try. We'll prove the $p = 2$ case later.)

The **unit ball** of a norm $\|\cdot\|$ on a vector space V is

$$B = \{x \in V : \|x\| \leq 1\} \tag{1.5}$$

A picture of the unit ball of a norm gives some geometric insight into the nature of the norm. Its especially illuminating to compare norms by comparing their unit balls. Lets consider the family of norms defined in (1.3) and (1.4) on R^2. The boundary of each unit ball is the curve $\|x\| = 1$ and clearly goes through the points $(1, 0)$ and $(0, 1)$. Its not hard to compute that the unit ball of $\|\cdot\|_\infty$ is a square with sides parallel to the coordinate axes. The unit ball of $\|\cdot\|_2$ is a disk contained in this square, and unit ball of $\|\cdot\|_1$ is a diamond shape thing contained in this disk. Less obvious, but reasonable, is the fact that the other unit balls lie inside the square and contain the diamond. Draw pictures.

Definition An *inner product* on a real or complex vector space V is a function on $V \times V$ to the scalars such that

1. $(x, x) \geq 0$ for all $x \in V$ and $(x, x) = 0$ only if $x = 0$.
2. $(ax + by, z) = a(x, z) + b(y, z)$ for all scalars a, b and all vectors x, y.
3. $(x, y) = \overline{(y, x)}$.

Examples

1.6 $V = R^n$ with $(x, y) = \sum_{j=1}^n x_j y_j$.

1.7 $V = \mathbb{C}^n$ with $(x, y) = \sum_1^n x_j \overline{y_j}$

1.8 $V = C([0, 1])$ with $(f, g) = \int_0^1 f(t)\overline{g(t)}dt$. [It would be good for you to verify yourself that this really is an inner product.]

1.9 $V = l^2$, which is the standard notation for the set of sequences $x = (a_1, a_2, \ldots)$ such that $\sum_{k=1}^\infty |a_k|^2 < \infty$. Define

$$(x, y) = \sum_{k=1}^\infty a_k \overline{b_k}$$

when $y = (b_1, b_2, \ldots)$. The series entering into this definition converges absolutely because of the identity $|a\bar{b}| \leq (|a|^2 + |b|^2)/2$. [It would be very good for you to verify that this really defines an inner product on l^2.]

Theorem 1.1 *(The Schwarz inequality.) In any inner product space*

$$|(x, y)| \le (x, x)^{1/2}(y, y)^{1/2} \tag{1.6}$$

H2

Proof: If x or y is zero then both sides are 0. So we can assume $x \neq 0$ and $y \neq 0$. In case the inner product space is complex choose $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $\alpha(x, y)$ is real. If the inner product space is real just take $\alpha = 1$. In either case let $p(t) = \|\alpha x + ty\|^2$ for all $t \in R$. Then

$$0 \le p(t) = (\alpha x + ty, \alpha x + ty) = \|x\|^2 + t^2\|y\|^2 + 2t(\alpha x, y)$$

because $(\alpha x, y) + (y, \alpha x) = 2Re(\alpha x, y) = 2(\alpha x, y)$. So, by the quadratic formula, the discriminant, $b^2 - 4ac \le 0$. That is , $4(\alpha x, y)^2 - 4\|x\|^2\|y\|^2 \le 0$. Thus$|\alpha(x, y)|^2 \le \|x\|^2\|y\|^2$. Now use $|\alpha| = 1$. QED.

We are going to show next that in an inner product space one can always produce a norm from the inner product by means of the definition

$$\|x\| = \sqrt{(x, x)}. \tag{1.7}$$

H3

Corollary 1.2 *Define* $\|\cdot\|$ *by* (H3 1.7). *Then*

$$\|x + y\| \le \|x\| + \|y\|.$$

Proof: Using (H2 1.6) we find

$$\begin{aligned}\|x + y\|^2 &= (x + y, x + y) = \|x\|^2 + \|y\|^2 + 2Re(x, y)\\ &\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.\end{aligned}$$

QED.

Corollary 1.3 $\|\cdot\|$ *is a norm.*

Proof: Using the previous corollary its easy to verify the three properties in the definition of a norm. QED

1.2 Bessel's inequality and orthonormal bases

Definition An *orthonormal* sequence in an inner product space is a set $\{e_1, e_2, \dots\}$ (which we allow to be finite or infinite) such that

$$(e_j, e_k) = 0 \;\; if \;\; j \neq k \; and = 1 \; if j = k$$

Lemma 1.4 *(Bessel's inequality) Let $e_1, e_2, \dots$ be an orthonormal set in a (real or complex) inner product space V. Then for any $x \in V$*

$$\|x\|^2 \geq \sum_{j=1}^{\infty} |(x, e_j)|^2$$

Proof: Let $a_k = (x, e_k)$. Then, for any integer n,

$$\begin{aligned}
0 &\leq (x - \sum_{k=1}^{n} a_k e_k, x - \sum_{k=1}^{n} a_k e_k) \\
&= \|x\|^2 - \sum_{k=1}^{n} [a_k (e_k, x) + (x, e_k)\overline{a_k}] + \sum_{j,k} a_j \overline{a_k} (e_j, e_k) \\
&= \|x\|^2 - \sum_{k=1}^{n} |a_k|^2 - \sum_{k=1}^{n} |a_k|^2 + \sum_{k=1}^{n} |a_k|^2 \\
&= \|x\|^2 - \sum_{k=1}^{n} |a_k|^2
\end{aligned}$$

So $\sum_{k=1}^{n} |a_k|^2 \leq \|x\|^2$ for all n. Now take the limit as $n \to \infty$. QED

Definition In a vector space with a given norm we define

$$\lim_{n \to \infty} x_n = x$$

to mean

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

We're all familiar with the concept of an orthonormal basis in finite dimensions: $e_1, \dots, e_n$ is an orthonormal basis of a finite dimensional inner product space V if

(a) the set $\{e_1, \dots, e_n\}$ is orthonormal and

(b) every vector $x \in V$ is a sum :$x = \sum_{j=1}^{n} a_j e_j$.

If V is infinite dimensional then we should expect that the concept of orthonormal basis should, similarly, be given by the requirement (a) (as before) and (b) every vector $x \in V$ is a sum

$$x = \sum_{j=1}^{\infty} a_j e_j. \tag{1.8}$$ H8

And this is right. Of course writing an infinite sum means, as usual, a limit of finite sums: $\lim_{n\to\infty} \|x - \sum_{j=1}^{n} a_j e_j\| = 0$. There is an unfortunate and downright annoying aspect to the equation (1.8), however. We see from Bessel's inequality that (1.8) implies that $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.

Suppose that we are given the orthonormal sequence $\{e_1, e_2, \dots\}$ and a sequence of (real or complex) numbers a_j such that $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. Does the series $\sum_{k=1}^{\infty} a_k e_k$ converge to some vector in V? If not can we really say then that we have "coordinatized" V if we don't even know which coordinate sequences $\{a_1, a_2, \dots\}$ actually correspond to vectors in V (by the formula (1.8))? Here is an example of how easily convergence of the series $\sum_{k=1}^{\infty} a_j e_j$ can fail, even when $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.

Let F be the subspace of l^2 consisting of finitely nonzero sequences. Thus $x \in F$ if $x = (a_1, \dots, a_n, 0, 0, 0, \dots)$. F is clearly a vector space and the inner product on l^2 restricts to an inner product on F. Let $e_1 = (1, 0, 0, 0, \dots), e_2 = (0, 1, 0, 0, 0, \dots)$, etc. The sequence $\{e_1, e_2, \dots\}$ is orthonormal. Let $a_j = 2^{-j}$. Then $\sum_{j=1}^{\infty} |a_j|^2 < \infty$. Now the sequence of partial sums, $x_n = \sum_1^n a_j e_j$ converges to the vector $x = \sum_1^{\infty} a_j e_j$ in l^2 (you verify this). But x is not in F. So there is no vector in F whose coodinates are the nice sequence $\{2^{-j}\}$. Of course we caused this trouble by making "holes" in l^2. These circumstances are analogous to the "holes" in the set Q of rational numbers. For example the sequence $s_n = \sum_{k=1}^{n} 1/k!$ is a sequence of rational numbers whose limit is e. But e is not rational. So there is a hole in Q at e. Question: Can you immagine how intolerably complicated calculus would be if we had to worry about these holes in Q? (E.g. $f'(x) = 0$ gives the maximum of F on Q provided x is rational!) The same nuisance would arise if we allowed holes when dealing with ON bases. We are going to eliminate holes!

Definition. A sequence $x_1, x_2, \dots$ in a normed vector space is a *Cauchy sequence* if

$$\lim_{n,m\to\infty} \|x_n - x_m\| = 0.$$

That is, for any $\epsilon > 0$ there is an integer N such that $\|x_n - x_m\| < \epsilon$ whenever n and $m \geq N$.

Remark. If x_n converges to a vector x in any normed space then the sequence is a Cauchy sequence. Proof: Same as proof of Proposition 9.3 in the Appendix. Just replace $|\cdot|$ by $\|\cdot\|$.

Definition. A normed vector space is *complete* if every Cauchy sequence in V has a limit in V. A *Banach* space is a normed vector space which is complete.

Examples. The spaces V in Examples 1.1 to 1.4 are complete.

[You're supposed to know this from first year calculus.]

In Example 1.5 the space $C([0,1])$ is complete in the norm $\|f\|_\infty$ but not in the norm $\|f\|_1$. [See if you can prove both of these statements.]

Definition. A *Hilbert space* is an inner product space which is complete in the associated norm (1.7).

Examples. The Examples 1.6, 1.7, 1.9 are Hilbert spaces. But Example (1.8) is not complete. The proof that Example 1.9 is complete will be given only on popular demand. It is extremely unfortunate that the Example 1.8 is not complete. In order to get a complete space one must throw in with the continuous functions all the functions whose square is integrable. This is such an important example that it gets its own notation.

Notation.

$$L^2(0,1) \text{ is the set of functions } f:(0,1) \to \mathbb{C} \text{ such that } \quad \int_0^1 |f(t)|^2 dt < \infty$$

For these functions we define

$$(f,g) = \int_0^1 f(t)\overline{g(t)}dt$$

This is an inner product on $L^2(0,1)$ (easy to verify) and the associated norm is

$$\|f\| = \sqrt{\int_0^1 |f(t)|^2 dt}$$

Of course if we wish to consider square integrable functions on some other set, such as R we would denote it by $L^2(R)$.

Just as the real numbers fill in the "holes" in the rational numbers so also one may view $L^2(0,1)$ as filling in the "holes" in $C([0,1])$. Here is the definition that makes this notion precise.

Definition. A subset A of a Hilbert space H is called *dense* if for any $x \in H$ and any $\epsilon > 0$ there is a vector y in A such that $\|x - y\| < \epsilon$. In words: you can get arbitrarily close to any vector in H with vectors in A. As we know, the rational numbers are dense in the real numbers. For example the rational number 3.141592650000000 is pretty close to π. Similarly, if you cut off the decimal expansion of a real number at the twentieth digit after the decimal point then you have a rational number which is very close to the given real number.

GOOD NEWS: $C([0,1])$ is dense in $L^2(0,1)$. You may use this fact whenever you find it convenient. Its often best to prove some formula for an easy to handle dense set first, and then show that it automatically extends to the whole Hilbert space. We'll see this later in the context of Fourier transforms.

Here is the first important consequence of completeness.

lem1.5 **Lemma 1.5** *Suppose that H is a Hilbert space and that $\{e_1, e_2, \dots\}$ is any ON sequence. Let c_k be any sequence of scalars such that $\sum_{k=1}^{\infty} |c_k|^2 < \infty$. Then the series*

$$\sum_{k=1}^{\infty} c_k e_k$$

converges to some unique vector x in H.

Proof: Let $s_n = \sum_{k=1}^{n} c_k e_k$. We must show that the sequence converges to a vector in H. Since we don't know in advance that there is a limit, x, to which the sequence converges we will show instead that the sequence is a Cauchy sequence. Suppose that $n > m$. Then $\|s_n - s_m\|^2 = \|\sum_{k=m+1}^{n} c_k e_k\|^2 = \sum_{k=m+1}^{n} |c_k|^2 \to 0$ as $m, n \to \infty$ because the series $\sum_{k=1}^{\infty} |c_k|^2$ converges. Now, because H is assumed to be complete, we know that there exists a vector x in H such that $\lim_{n\to\infty} s_n = x$. Since limits are unique, x is unique. (You prove this on the way to your next class.) Of course we will write

$$x = \sum_{k=1}^{\infty} c_k e_k, \tag{1.9}$$

keeping in mind that this means that x is the limit of the finite sums. QED.

Lemma 1.6 *In any inner product space the function $x \to (x, y)$ is continuous for each fixed element y.*

Proof. If $x_n \to x$ then $|(x_n, y) - (x, y)| = |(x_n - x, y)| \leq \|x_n - x\|\|y\| \to 0$. QED

Of course (x, y) is also a continuous function of y for each fixed x. One can either repeat the preceding proof or just use $\overline{(y, x)} = (x, y)$.

thmH.5 **Theorem 1.7** *Let $e_1, e_2, \ldots$ be an orthonormal sequence in a (real or complex) Hilbert space H. Then the following are equivalent.*

a. $e_1, e_2, \ldots$ is a maximal ON set. That is, it is not properly contained in any other ON set.

b. For every vector $x \in H$ we have

$$x = \sum_{k=1}^{\infty} a_k e_k \qquad \text{where} \;\; a_k = (x, e_k)$$

c. For every pair of vectors x and y in H we have

$$(x, y) = \sum_{k=1}^{\infty} a_k \overline{b_k} \;\; \text{where } a_k = (x, e_k) \;\; \text{and } b_k = (y, e_k)$$

d. For every vector x in H we have

$$\|x\|^2 = \sum_{k=1}^{\infty} |a_k|^2$$

Proof: We will show that a. $\implies$ b. $\implies$ c. $\implies$ d. $\implies$ a.

Assume that a. holds. Let $x \in H$. By Bessel we have $\sum_1^\infty |a_k|^2 < \infty$. By Lemma lem1.5 1.5, $y \equiv \sum_1^\infty a_k e_k$ exists. But $(x-y, e_j) = a_j - \lim_{n\to\infty}(\sum_{k=1}^n a_k e_k, e_j) = a_j - a_j = 0$ for all j. If $x \neq y$ then let $h = (x-y)/\|x-y\|$. One can now adjoin h to the original set and obtain a larger ON set. So we must have $x - y = 0$. This proves that b. holds.

Assume now that b. holds. Then

$$(x, y) = \lim_{n\to\infty}(\sum_{j=1}^n a_j e_j, y) = \lim_{n\to\infty}\lim_{m\to\infty}(\sum_{j=1}^n a_j e_j, \sum_{k=1}^m b_k e_k)$$
$$= \lim_{n\to\infty}\lim_{m\to\infty}\sum_{j=1}^{min\ n,m} a_j \overline{b_j} = \sum_{j=1}^{\infty} a_j \overline{b_j}.$$

So c. holds.

Next, assume that c. holds. Put $y = x$ to derive that d. holds.

Finally, assume that d. holds. If $e_1, e_2, \ldots$ is not a maximal ON set then there exists a vector $x \neq 0$ such that $(x, e_k) = 0$ for all k. So the "coordinates", $a_k = (x, e_k)$ are all zero. But from d. we see that $\|x\|^2 = \sum_{k=1}^\infty |a_k|^2 = 0$. So $x = 0$. Contradiction. QED.

Now we're ready for the definition of ON basis.

Definition. An ON sequence $\{e_1, e_2, \ldots\}$ in a Hilbert space H is an *ON basis* of H if condition b. in Theorem 1.7 holds.

Of course we could have used any of the other conditions in Theorem 1.7 for the definition of ON basis because they're equivalent. So why did I use condition b. for the definition? Because surveys of your predecessors show that it's the most popular.

1.3 Problems on Hilbert space

1. Let $f_1, f_2, \ldots, f_9$ be an orthonormal set in $L^2(0,1)$. Assume that

(A) $\int_0^1 6x f_1(x)dx = 2$ and

(B) $\int_0^1 6x f_2(x)dx = 2\sqrt{2}$

What can you say about the value of

$$\int_0^1 6x f_5(x)dx?$$

Give reasons.

2. Let

$$u_1(x) = 1/\sqrt{2}, \ -1 \le x \le 1$$

and

$$u_2(x) = \sqrt{\frac{3}{2}}x, \ -1 \le x \le 1.$$

Suppose that f and g are in $L^2([-1,1])$ and $\|f - g\| \le 5$. Let $a_j = \int_{-1}^1 u_j(x)\overline{f(x)}dx$, $j = 1,2$ and $b_j = \int_{-1}^1 u_j(x)\overline{g(x)}dx$. Show that $\sum_{j=1}^2 |a_j - b_j|^2 \le 25$. Cite any theorem you use.

3. Suppose that $f : [-1,1] \to R$ satisfies

$$\int_{-1}^1 |f(x)|^2 dx = 21$$

and

$$\int_{-1}^1 f(x)dx = 6.$$

What can you say about the size of $\int_{-1}^1 xf(x)dx$?

4. Suppose that u_1, u_2 are O.N. vectors in an inner product space H. Let $f \in H$ and assume that

$$\|f\|^2 = |a_1|^2 + |a_2|^2$$

where $a_j = (f, u_j)$ for $j = 1,2$. Show that $f = a_1u_1 + a_2u_2$.

5. The Hermite polynomials are the sequence of polynomials $H_n(x)$ uniquely determined by the properties:

a) $H_n(x)$ is a real polynomial of degree n, $n = 0, 1, 2, \ldots$ with positive leading coefficient.

b) The functions

$$u_n(x) = H_n(x)e^{-x^2/4}$$

form an O.N. sequence in $L^2(R)$.

Fact that you may use: If f is in $L^2(R)$ and $\int_{-\infty}^{\infty} f(x)u_n(x)dx = 0$ for $n = 0, 1, 2, \ldots$ then $f = 0$.

Let $c_n = \int_{-\infty}^{\infty} e^{-|x|}u_n(x)dx$.

Evaluate $\sum_{n=0}^{\infty} c_n^2$.

6. Let $\{f_1, f_2, \ldots\}$ be an O.N. set in a Hilbert space H. Prove that it is an O.N. basis if and only if the finite linear combinations

$$\sum_{j=1}^{n} a_j f_j \text{ (n finite but arbitrary)}$$

are dense in H.

7. Let $\{f_1, f_2, \ldots\}$ be an O.N. set in a Hilbert space H. Prove that it is an O.N. basis if and only if Parseval's equality

$$\|g\|^2 = \sum_{j=1}^{\infty} |(f_j, g)|^2$$

holds for a dense set of g in H.

8. Suppose that $\{f_1, f_2, \ldots\}$ is an O.N. basis of a Hilbert space H and that $\{g_1, g_2, \ldots\}$ is an O.N. sequence in H. Suppose further that

$$\sum_{j=1}^{\infty} \|g_j - f_j\|^2 < 1.$$

Prove that $\{g_1, g_2, \ldots\}$ is also an O.N. basis.

Hint: Use Theorem 1.7 by supposing that there exists $h \neq 0$ such that $(h, g_j) = 0$ for all j.

2 Second order linear ordinary differential equations

2.1 Introduction

Consider the differential equation

$$u''(x) - u(x) = f(x) \tag{2.1}$$

on the interval $0 \leq x \leq 1$. Here f is a given function on this interval. As you know, the solution to a second order ordinary differential equation requires specification of some additional information in order for the ODE to pick out a solution uniquely. For example one might specify $u(0) = 3$ and $u'(0) = 5$ to pick out a unique solution to (2.1). This is the *initial value problem*: both pieces of information are specified at the same endpoint. We are going to be primarily concerned with the *boundary value problem*: the two pieces of information will be specified at opposite endpoints. Let's focus on the boundary value problem for the differential equation (2.1) given by the boundary data

$$u(0) = 0, \quad u(1) = 0. \tag{2.2}$$

Our goal is to show that the solution to (2.1) and (2.2) can be represented in the form

$$u(x) = \int_0^1 G(x, y) f(y) dy \tag{2.3}$$

for some function $G(x, y)$. G is called the **Green function** for the boundary value problem (2.1), (2.2). Here is the method for constructing the Green function in this simple example.

STEP 1. Consider first the homogeneous version of (2.1), namely

$$u'' - u = 0 \tag{2.4}$$

Since this equation has constant coefficients its easy to write down the general solution. It is

$$u(x) = A \sinh x + B \cosh x$$

We will need first to find the solutions which satisfies the LEFT HAND boundary condition $u(0) = 0$. These are clearly given by choosing $B = 0$.

We will see later that our procedure doesn't care what A is. So lets just choose $A = 1$. Denote the solution that we've now got by U. So

$$U(x) = \sinh x$$

STEP 2. Do the same thing all over again, but using the RIGHT HAND boundary condition. To this end we may write the general solution to (2.4) 04 in the form

$$u(x) = C \sinh(x - 1) + D \cosh(x - 1)$$

A solution that satisfies the right hand boundary condition is

$$V(x) = \sinh(x - 1)$$

(and this is the only one, up to a scalar multiple.)

STEP 3. Define

$$W(x) = det \begin{pmatrix} U(x) & U'(x) \\ V(x) & V'(x) \end{pmatrix}$$

Inserting our particular U and V we can compute that

$$W(x) = \sinh 1 \quad 0 \le x \le 1 \tag{2.5}$$ 05

STEP 4. Define

$$G(x, y) = \begin{cases} \frac{U(x)V(y)}{W(y)}, & \text{for } 0 \le x \le y \le 1 \\ \frac{U(y)V(x)}{W(y)}, & \text{for } 0 \le y \le x \le 1. \end{cases} \tag{2.6}$$ 06

In our case this expression reduces to

$$G(x, y) = \begin{cases} \frac{(\sinh x)\sinh(y-1)}{\sinh 1}, & \text{for } 0 \le x \le y \le 1 \\ \frac{(\sinh y)\sinh(x-1)}{\sinh 1}, & \text{for } 0 \le y \le x \le 1. \end{cases}$$

Notice that on the overlapping portion of these definitions, namely on the diagonal $x = y$, the two halves of these formulas agree. So the function $G(x, y)$ is well defined even on the diagonal of the square $[0, 1] \times [0, 1]$. Moreover, for each x, $G(x, y)$ is a continuous function of y and for each y $G(x, y)$ is a continuous function of x because the limits from the left and right (or up to down) agree on the diagonal.

NOW let

$$u(x) = \int_0^1 G(x, y) f(y) dy \tag{2.7}$$ 08

for ANY continuous function f.

I claim that u is a solution to the boundary value problem (2.1), (2.2). The easy part of this is seeing that the function u in (2.7) satisfies the two boundary conditions (2.2). Indeed the definition of G shows that $G(0, y) = 0$ for all y and $G(1, y) = 0$ for all y. This is why we chose U and V as we did. So u is zero at the two boundaries. To prove that u satisfies (2.1) we must differentiate the right side of (2.7) a couple of times. The naive reader (not you) might think that we can just put d^2/dx^2 under the integral sign. But if you do this you will find $(d^2/dx^2)G(x, y) - G(x, y) = 0$ on each half of the square because both $U(x)$ and $V(x)$ satisfies this equation. This would then give $u'' - u = 0$, which is not what we want. So here is what we must do. There is trouble with $G(x, y)$ as x passes through y. Although $G(x, y)$ is continuous in x for fixed y its first derivative, $G_x(x, y)$, actually has a jump as x passes y, as we will see in a moment. So $G_{xx}(x, y)$ doesn't even exist at $x = y$. To deal with this we just put the trouble spot into the limits of integration thus:

$$u(x) = \int_0^x G(x, y)f(y)dy + \int_x^1 G(x, y)f(y)dy.$$

In each integral the integrand is a nice function of x, provided y is where the limits of integration allow it to go. There is never a crossing over the diagonal. Here is the computation. As you know there is a contribution to the derivative of u from the x that occurs in the limits of integration.

$$u'(x) = G(x, x^-)f(x) - G(x, x^+)f(x) \tag{2.8}$$

$$+ \int_0^x G_x(x, y)f(y)dy + \int_x^1 G_x(x, y)f(y)dy \tag{2.9}$$

In the first line the expression x^- indicates that we are letting y approach x from below, as it should, while x^+ indicates that y is approaching x from above. But now recall that you admitted a while back that the function $y \mapsto G(x, y)$ is continuous. Therefore the first line is zero. So

$$u'(x) = \int_0^x G_x(x, y)f(y)dy + \int_x^1 G_x(x, y)f(y)dy, \tag{2.10}$$ 011

which is just what we would have gotten if we had simply put d/dx under the integral in (2.7). But the next time we won't be so lucky. We're going to repeat this procedure for the second derivative. But we'll see that the "first line" is not zero this time. We find from (2.10)

$$\begin{aligned} u''(x) &= G_x(x,x^-)f(x) - G_x(x,x^+)f(x) \\ &\quad + \int_0^x G_{xx}(x,y)f(y)dy + \int_x^1 G_{xx}(x,y)f(y)dy \end{aligned} \tag{2.11}$$ 012

This time the first line does not give zero because $G_x(x,y)$ has a jump as y passes by x. How much is the jump? Just look at the definition of G and the definition of W: $G_x(x,y) = U(y)V'(x)/W(y)$ if $y < x$. So $G_x(x,x^-) = U(x)V'(x)/W(x)$. Similarly $G_x(x,x^+) = U'(x)V(x)/W(x)$. So

$$\begin{aligned} G_x(x,x^-) - G_x(x,x^+) &= \frac{U(x)V'(x) - U'(x)V(x)}{W(x)} \\ &= 1. \end{aligned} \tag{2.12}$$ 013

Ha! So the first line in (2.11) is just $f(x)$. But we have already observed that $G_{xx}(x,y) = G(x,y)$ (at least when $y \neq x$, and this is where y is ranging in the two integrals.) Putting this together we now find that $u''(x) = f(x) + u(x)$, which is exactly the equation (2.1). Thus $u(x)$, as given by (2.7), satisfies both (2.1) and (2.2). So $G(x,y)$ is the Green function for the boundary value problem (2.1),(2.2).

Our aim in this section is to carry out this procedure for a general second order ordinary differential equation. To this end we must be able to construct the functions U and V and understand the function W. All of these are issues pertaining to the initial value problem, which is much easier to deal with than the boundary value problem and whose basic theory we will review in the next section.

But before leaving this successful example lets rewrite **informally** the preceeding computations. We actually showed that

$$(d^2/dx^2 - 1)\int_0^1 G(x,y)f(y)dy = f(x) \tag{2.13}$$ 018

If we (innocently) rewrite this as

$$\int_0^1 (d^2/dx^2 - 1)G(x,y)f(y)dy = f(x)$$

then we could claim that we have shown that

$$(d^2/dx^2 - 1)G(x, y) = \delta(x - y). \qquad (2.14)$$ 020

And what does (2.14) mean? It means exactly that (2.13) holds, which we have proved. So if you are fond of δ functions (and who isn't) then all is well.

In truth, this neat approach to Green functions, as embodied in the fundamental formula (2.6), can break down for a general second order differential equation. We need to see how this can happen before proceeding to a general second order ODE. Here is a simple example of

BREAK DOWN

Lets modify the equation (2.1) slightly. Consider the equation

$$u''(x) + \pi^2 u(x) = f(x) \quad 0 \le x \le 1. \qquad (2.15)$$ 021

The general solution to the corresponding homogeneous equation $u'' + \pi^2 u = 0$ is $u(x) = A\sin(\pi x) + B\cos(\pi x)$. Since $\cos 0 \neq 0$ and $\cos\pi \neq 0$ the desired function U is $U(x) = sin(\pi x)$ and the desired function V is ALSO $V(x) = \sin(\pi x)$. Consequently $W(x) = 0$ for all x.!! The formula (2.6) is now meaningless.!!! That's life. So is there a Green function? Answer: No. It will be important for us to understand exactly why and when such failure occurs.

2.2 The initial value problem.

We consider a closed bounded interval $[a, b]$. We wish to study several problems associated with the differential operator

$$(Lu)(x) = p_2(x)u''(x) + p_1(x)u'(x) + p_0(x)u(x). \qquad (2.16)$$ 1.1

We will assume throughout that the coefficients p_j are continuous on $[a, b]$ and that $p_2(x) > 0$ on the entire closed interval.

thm01 **Theorem 2.1** *(Existence and uniqueness for the initial value problem)*
Let f be a continuous function on $[a, b]$ and let c and c' be two real numbers. Then there exists a unique function $u \in C^2([a, b])$ satisfying

$$Lu(x) = f(x) \ \text{ for all } x \in [a, b] \text{ and } u(a) = c \text{ and } u'(a) = c'. \qquad (2.17)$$ 1.2

Proof: Use Picard's method. See e.g. Kreyszig "Advanced Engineering Mathematics"

defWronsk **Definition 2.2** *(Wronskian) Let f and g be two functions in $C^1([a,b])$. The Wronskian of f and g is the function*

$$W(f,g)(x) = \det \begin{pmatrix} f(x) & f'(x) \\ g(x) & g'(x) \end{pmatrix} \tag{2.18}$$ 1.3

Recall that f and g are said to be linearly dependent on $[a,b]$ if there are constants α and β such that

$$\alpha f(x) + \beta g(x) \equiv 0$$

In this case we may differentiate this equation and find that

$$\alpha f'(x) + \beta g'(x) \equiv 0$$

when f and g are differentiable. This shows that the rows of the matrix in (2.18) are linearly dependent for every $x \in [a,b]$. Hence $W(f,g) \equiv 0$. This proves

lem03 **Lemma 2.3** *If f and g are in $C^1([a,b])$ and are linearly dependent on the interval $[a,b]$ then*

$$W(f,g)(x) = 0 \text{ for all } x \in [a,b].$$

The partial converse of this is a little bit more subtle.

Lemma 2.4 *Let u and v be two solutions of $Lu = 0$. Then u and v are linearly independent on $[a,b]$ if and only if $W(u,v)$ is nowhere zero.*

Proof: Assume $Lu = Lv = 0$. If, for some point x_0 one has $W(u,v)(x_0) = 0$ then the points $(u(x_0), u'(x_0))$ and $(v(x_0), v'(x_0))$ in the plane are linearly dependent vectors in $\mathbb{R}^2$. Therefore there exist nonzero constants α, β such that

$$\alpha u(x_0) + \beta v(x_0) = 0 \text{ and } \alpha u'(x_0) + \beta v'(x_0) = 0.$$

Let $g(x) = \alpha u(x) + \beta v(x)$. Then $Lg = 0$. But $g(x_0) = 0$ and $g'(x_0) = 0$. Therefore $g(x) = 0$ for all $x \in [a,b]$ by the uniqueness portion of Theorem 2.1. Hence u and v are linearly dependent on $[a,b]$. Conversely, if u and v are linearly dependent on $[a,b]$ then Lemma 2.3 shows that $W(u,v)$ is identically zero (even if u and v are not solutions to $Lu = 0$.)

2.3 The boundary value problem and Green functions

thm02 **Theorem 2.5** *Let*

$$L(u) = p_2(x)u'' + p_1(x)u' + p_0(x)u \tag{2.19}$$
$$A(u) = \alpha u(a) + \alpha' u'(a) \tag{2.20}$$
$$B(u) = \beta u(b) + \beta' u'(b) \tag{2.21}$$

for $u \in C^2([a,b])$*, where* p_0*,* p_1*,* p_2 *are in* $C([a,b])$*,* α*,* α'*,* β*,* β' *are real numbers, and* $p_2 > 0$ *on* $[a,b]$*. Then the inhomogenous system*

1) $L(w) = f \qquad \in C([a,b])$
$A(w) = \alpha_1$
$B(w) = \beta_1$

has a solution for all f*,* α_1*,* β_1 *if and only if the homogeneous system*

2) $L(u) = 0$
$A(u) = 0$
$B(u) = 0$

has no non–zero solution.

Proof: Define functions U, V, and F with the aid of the basic existence and uniqueness theorem, Theorem thm01 2.1, so as to satisfy

$$L(U) = 0, \quad U(a) = \alpha', \quad U'(a) = -\alpha \tag{2.22}$$
$$L(V) = 0, \quad V(b) = \beta', \quad V'(b) = -\beta \tag{2.23}$$
$$L(F) = f, \quad F(a) = F'(a) = 0 \tag{2.24}$$

Then clearly $A(U) = 0$ and $B(V) = 0$.

Thus for Dirichlet boundary conditions $[\alpha = 1, \alpha' = 0, \beta = 1, \beta' = 0]$, U and V look like this

Let

$$w = cU + dV + F.$$

Then

$$L(w) = f \tag{2.25}$$
$$A(w) = dA(V) \tag{2.26}$$
$$B(w) = cB(U) \tag{2.27}$$

Now assume that the system 2) has no non–zero solutions. Since $L(U) = A(U) = 0$ it then follows that $B(U) \neq 0$. Similarly $A(V) \neq 0$. Hence for any α_1, β_1 we may choose c and d so that w satisfies 1).

Conversely assume that 1) has a solution for all f_1, α_1, β_1 and that 2) has a solution u not identically zero. Then there exists a function v such that

$$L(v) = 0 \tag{2.28}$$
$$A(v) = 0 \tag{2.29}$$
$$B(v) = 1 \tag{2.30}$$

But $(u(a), u'(a)) \in R^2$ is orthogonal to (α, α') as is also $(v(a), v'(a))$ and neither is zero — by uniqueness theorem. Hence one is a multiple of the other. Say $c(u(a), u'(a)) = (v(a), v'(a))$. Then the function $cu - v$ has zero initial data at a and satisfies $L(cu - v) = 0$. $cu - v = 0$ on $[a, b]$. But $B(u) = 0$. $B(v) = 0$, contradiction.

Corollary 2.6 *If the system 2) has no non zero solution then*

a) the functions U and V constructed in the previous proof are linearly independent and

b) their Wronskian W is nowhere zero.

Proof: It suffices to show U and V are linearly independent. If they aren't then there exists a constant c such that $U = cV$. But then $B(U) = cB(V) = 0$. So U satisfies 2). But U is not identically zero on $[a, b]$.

thm03 **Theorem 2.7** *(Green function.) Assume that the system 2) has no non–zero solution. Then the system*

$$L(u) = f$$
$$A(u) = 0$$
$$B(u) = 0 \tag{2.31}$$

2.1

has the unique solution

$$u(x) = \int_a^b G(x,y) f(y) dy \tag{2.32}$$ 2.2

where

$$G(x,y) = \begin{cases} U(x)V(y)/(p_2(y)W(y)) & a \le x \le y \le b \\ U(y)V(x)/(p_2(y)W(y)) & a \le y \le x \le b \end{cases} \tag{2.33}$$ 2.3

and

$$W = UV' - VU'. \tag{2.34}$$ 2.4

Proof. As in the introduction to this section we are going to have to break up the integral into two parts to take into account the jump in the derivative of G. Thus we will write

$$\begin{aligned} u(x) &= \int_a^b G(x,y) f(y) dy && (2.35) \\ &= \int_a^x G(x,y) f(y) dy + \int_x^b G(x,y) f(y) dy. && (2.36) \end{aligned}$$ 2.5

The derivative of each integral with respect to x will have two terms, corresponding to the two appearances of x. Differentiating first with respect to the x that appears in the limits of the integrals the fundamental theorm of calculus gives

$$\begin{aligned} u'(x) &= G(x,x^-) f(x) - G(x,x^+) f(x) && (2.37) \\ &+ \int_a^x G_x(x,y) f(y) dy + \int_x^b G_x(x,y) f(y) dy. && (2.38) \end{aligned}$$

Thus the contribution to $u'(x)$ from the limits cancel because $G(x,y)$ is continuous in y at $y = x$. We therefore find

$$u'(x) = \int_a^x G_x(x,y) f(y) dy + \int_x^b G_x(x,y) f(y) dy. \tag{2.39}$$ 2.6

We need now the key identity that describes the jump of the first derivative of G. It is

$$G_x(x,x^-) - G_x(x,x^+) = \frac{1}{p_2(x)} \tag{2.40}$$ 2.6.1

This may be derived from (2.33). Just imitate the derivation of (2.12). Now differentiate (2.39) to find

$$u''(x) = G_x(x, x^-)f(x) - G_x(x, x^+)f(x) \tag{2.41}$$

$$+ \int_a^x G_{xx}(x, y)f(y)dy + \int_x^b G_{xx}(x, y)(y)f(y)dy. \tag{2.42}$$

$$= f(x)/p_2(x) + \int_a^x G_{xx}(x, y)f(y)dy + \int_x^b G_{xx}(x, y)(y)f(y)dy. \tag{2.43}$$

Hence multiplying by $p_2(x)$ we find

$$p_2(x)u''(x) = f(x) \tag{2.44}$$

$$+ \int_a^x p_2(x)G_{xx}(x, y)f(y)dy + \int_x^b p_2(x)G_{xx}(x, y)f(y)dy. \tag{2.45}$$

2.7

Now multiply (2.36) by $p_0(x)$ and multiply (2.39) by $p_1(x)$ and add them to (2.45) to find

$$Lu(x) = f(x) + \int_a^x (L_x G(x, y))f(y)dy + \int_x^b (L_x G(x, y))f(y)dy \tag{2.46}$$

$$= f(x) \tag{2.47}$$

because $L_x G(x, y) = 0$ off the diagonal. Finally, just as in the simple example in the introduction, the reader may verify that, for each $y \in (a, b)$, the function $x \mapsto G(x, y)$ satisfies the boundary conditions in (2.31). Hence so does u, by (2.36) and (2.39). ■

Definition 2.8 *(Eigenvalue.) A complex number λ is called an eigenvalue of the differential operator L with boundary conditions given by A and B if there exists a function u in $C^2([a, b])$ such that*

$$Lu = \lambda u \tag{2.48}$$

$$A(u) = 0 \tag{2.49}$$

$$B(u) = 0. \tag{2.50}$$

The function u is called an eigenfunction with eigenvalue λ.

Definition 2.9 *The differential operator L is called symmetric on $[a, b]$ if*

$$(Lu, v) = (u, Lv)$$

for all u and v in $C^2([a, b])$ which are zero in a neighborhood of each endpoint.

Fact: If L is symmetric then L **must have** the form given in the next theorem (wherein we change notational and sign conventions, taking $p_2(x) = -p(x) < 0$.) (Give a little try to find a proof of this fact by yourself.)

thm05 **Theorem 2.10** *Let p and q be continuous functions on $[a,b]$ with $p(x) > 0$ for all $x \in [a,b]$. Let*

$$Lu = -\Big[\frac{d}{dx}(p(x)u'(x)) - q(x)u(x)\Big] \tag{2.51}$$

$$A(u) = \alpha u(a) + \alpha' u'(a) \tag{2.52}$$

$$B(u) = \beta u(b) + \beta' u'(b). \tag{2.53}$$

Put

$$(u,v) = \int_a^b u(x)v(x)dx.$$

Then $(Lu,v) = (u,Lv)$ if u and $v \in C^2([a,b])$ and $A(u) = B(u) = A(v) = B(v) = 0$.

Moreover there exists a strictly increasing sequence λ_n of real eigenvalues for the operator L with boundary conditions $A(u) = 0$, $B(u) = 0$. In fact one has $\lim \lambda_n = \infty$. The corresponding eigenfunctions u_n may be chosen real. Assume they are normalized:

$$\int_a^b |u_n(x)|^2 dx = 1.$$

Then the sequence $u_1, u_2, \ldots$ forms an O. N. basis of $L^2(a,b)$.

2.4 Eigenfunction expansion of the Green function

We are going to use the notation

$$(Ku)(x) = \int_a^b K(x,y)u(y)dy$$

whenever $K(x,y)$ is a function on the square $[a,b] \times [a,b]$. The operator $u \mapsto Ku$ is called an integral operator. You can see from the repeated use of such expressions in the previous sections that this is a useful notation.

Let

$$Lu = -\frac{d}{dx}\Big(p(x)\frac{du}{dx}\Big) + q(x)u(x) \tag{2.54}$$

$$A(u) \text{ and } B(u) \text{ as usual.} \tag{2.55}$$

If λ is not an eigenvalue of L for the boundary conditions $A = 0$, $B = 0$ then $(L-\lambda)^{-1}$ exists and is given by a Green function $G_\lambda(x,\xi)$:

$$(L-\lambda)u = f \Rightarrow u = \int_a^b G_\lambda(x,\xi)f(\xi)d\xi.$$

Let $\lambda_1, \lambda_2, \dots,$ be the eigenvalues of L and $u_1, u_2, \dots$ the corresponding normalized (real) eigenfunction. Then

thm06 **Theorem 2.11**

$$G_\lambda(x,y) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j - \lambda} u_j(x)u_j(y).$$

Informal Proof: We assume all series converge and all interchanges are legal.

Let

$$K_\lambda(x,y) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j - \lambda} u_j(x)u_j(y).$$

Now

$$\int_a^b K_\lambda(x,y)u_n(y)dy = \sum_{j=1}^{\infty} \frac{1}{\lambda_j - \lambda}\int_a^b u(j(x)u_j(y)u_n(y)dy \tag{2.56}$$

$$= \frac{1}{\lambda_n - \lambda}u_n(x). \tag{2.57}$$

because all the other terms are zero. We may write this as

$$K_\lambda u_n = \frac{1}{\lambda_n - \lambda}u_n$$

Therefore

$$(L-\lambda)(K_\lambda u_n) = u_n$$

But $(L-\lambda)G_\lambda = Identity$. So

$$(L-\lambda)K_\lambda u_n = u_n = (L-\lambda)G_\lambda u_n.$$

Therefore $K_\lambda u_n = G_\lambda u_n$ for all n because $L-\lambda$ has trivial nullspace. Thus, for each x

$$(K_\lambda(x,\cdot), u_n) = (G_\lambda(x,\cdot), u_n) \quad \forall n$$

Therefore

$$K_\lambda(x,y) = G_\lambda(x,y).$$

This "proves" Theorem thm06 2.11. "QED"

2.5 Problems on ordinary differential equations

1. Apply Picard's iteration method to the following initial value problem. Find $y_1(x), y_2(x), y_3(x)$.

$$y' = 1 + xy, \quad y(0) = 1$$

Ref: Kreyszig, Advanced Engineering Mathematics (sec.1.11 in 3rd edition)

2. Determine which of the following operators are symmetric on the interval $[1, 5]$

a) $Lu = \dfrac{d^2u}{dx^2} + 3\dfrac{du}{dx} + 5u.$

b) $Lu = \dfrac{d^2u}{dx^2} + 4u.$

c) $Lu = \dfrac{d^2u}{dx^2} + 9\pi^2 u.$

3. Determine which of the operators in Problem 2 have inverses for the boundary conditions $u'(1) = u'(5) = 0$ and for these operators find the Green functions.

4. Let

$$Lu = x^2u'' + 2xu' \quad \text{on} \quad [1, 3].$$

a) Show that L is *symmetric* on this interval.

b) Find the *Green function* for L under Dirichlet boundary conditions: $u(1) = u(3) = 0$.

c) For the same boundary conditions find the *eigenfunctions* and *eigenvalues* of L.

Hint: Equations of the form $ax^2u'' + bxu' + cu = 0$ tend to have two linearly independent solutions of the form $u = x^\alpha$, α complex. When only one can be found in this form then $x^\alpha \ln x$ will give another one.

5. Prove the Fact preceding Theorem thm05 2.10.

3 Generalized functions. (δ functions and all that.)

3.1 Dual spaces

The concept of a dual space arises naturally in differential geometry, mechanics and general relativity. And we will need it later to understand generalized functions.

Definition 5.1 Let V be a real or complex vector space. A function $L : V \to$ scalars is called a *linear functional* if

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$$

for all x and y in V and for all scalars α and β. In other words a linear functional is a linear transformation from V into $\mathbb{R}$ (or $\mathbb{C}$ if V is a complex vector space.) The **dual space** to V is the set, denoted V^*, of all linear functionals on V.

For example the function which is identically 0 is a linear functional. (Check this against the definition.) Moreover if L_1 and L_2 are linear functionals then so is the function $aL_1 + bL_2$ for any scalars a and b. (Check this against the definition now!) Therefore V^* is itself a vector space. Its a new vector space constructed from the old one.

Example 5.2. Denote by $\mathcal{P}_3$ the vector space consisiting of polynomials of degree less or equal to 3. This is a four dimensional vector space because $\{1, t, t^2, t^3\}$ constitutes a basis. Here are some linear functionals on this space.

1. $p \mapsto L_1(p) = p(7)$.
2. $p \mapsto L_2(p) = \int_0^1 p(t)dt$.
3. $p \mapsto L_3(p) = \int_0^5 p(t) \sin t dt$.

[It would be best if you, personally, verify that each of these functions on $\mathcal{P}_3$ are linear functionals.]

The thing to take away from these examples is that there is no resemblance between V and V^*: you cannot really "identify" any of these three linear functionals on $\mathcal{P}_3$ with elements of $\mathcal{P}_3$ itself. RIGHT? V^* is really a different vector space from V itself. We have constucted a new vector space from the given one. This being the case, you have to regard the following theorem as remarkable.

Theorem 5.3 If V is an n-dimensional vector space then so is V^*.

Proof. Let $e_1, ...e_n$ be any basis of V. Then any vector $x \in V$ can be uniquely written

$$x = \sum_{j=1}^{n} a_j e_j \tag{3.1}$$ 5.1

Uniqueness means that each a_j is a function of x. Define

$$L_j(x) = a_j, \quad j = 1, ..., n.$$

Its straightforward to check that each function L_j is a linear functional. We will show that they form a basis of V^*.

1. They are linearly independent. Proof: Suppose that $M := \sum_{j=1}^{n} c_j L_j = 0$. Then $0 = M(e_k) = \sum_{j=1}^{n} c_j L_j(e_k) = \sum_{j=1}^{n} c_j \delta_{jk} = c_k$. So all the coefficients c_k are zero. Hence the functionals L_j are linearly independent.

2. They span V^*. Proof: Let L be any linear functional. Define $c_k = L(e_k)$. Claim: Then $L = \sum_{k=1}^{n} c_k L_k$. You can check this yourself by showing that both sides of this equation agree on each e_j and therefore on all linear combinations of the e_j. Thus they agree on all of V.

So we have now produced a basis of V^* consisting of n elements. Hence $dim V^* = n$. QED.

Terminology:5.4 The basis L_j described in the preceding proof is called the *dual basis* to the basis $e_1, ..., e_n$. It has the nice property that

$$L_j(e_k) = \delta_{jk}$$

Philosophic considerations 5.5. Having chosen the basis $e_1, ..., e_n$ of V we see that we automatically get a basis $L_1, ..., L_n$ of V^*. Since any vector x in V can be written uniquely in the form (5.1) we can now define a vector L_x in V^* by the formula

$$L_x = \sum_{j=1}^{n} a_j L_j$$

In this way we get a map $x \mapsto L_x$ from V onto V^*. You can check easily that this map is a) linear, b) one-to-one and c) onto V^*. This is, as some people would say, an isomorphism from V onto V^*. With the help of this map we could, if we wished, identify V and V^* and even go so far as to say that V and V^* are the "same" space. But there is a catch: A *choice* of basis has been made in constructing this isomorphism. If Jim goes into one room and chooses a basis $e_1, ..., e_n$ to construct this isomorphism and Jane goes into another room and chooses a basis the chances are that they will choose different bases. Then they will arrive at different isomorphisms. So each one will identify V with V^* in different ways. Jim will say that the vector $x \in V$ corresponds to a certain linear functional L and Jane will say, no, it corresponds to a different linear functional, M.

When an isomorphism between two vector spaces depends on someone's choice of a basis we say that the isomorphism is not natural. If you should nevertheless decide to think of these two vector spaces as the "same" (i.e. identify them) then sooner or later you will run into conceptual and even computational trouble.

But there is an important circumstance in which one really can justify identifying V and V^*. (Some readers might recognize the next theorem as "raising and lowering" indices.)

Theorem 5.6. Suppose that V is a real finite dimensional vector space and $(\cdot, \cdot)$ is a given inner product on V. Then for any linear functional L on V there is a unique vector y in V such that

$$L(x) = (x, y) \quad \text{for all} \quad x \in V.$$

Denote by L_y the linear functional determined by y in this way. That is,

$$L_y(x) = (x, y) \quad \text{for all } x \in V. \tag{3.2}$$ 5.10

Then the map

$$y \mapsto L_y$$

is a one-to-one linear map of V onto V^*. (I.e. it is an isomorphism.)

Proof. The map $y \mapsto L_y$ is clearly linear. (You better check this. It will be good practice in dealing with these structures.) Moreover this map is one-to-one because if $L_y = 0$ then in particular $L_y(y) = 0$. That is, $(y, y) = 0$. So $y = 0$. Therefore the map $y \mapsto L_y$ is one-to-one. Hence, by the rank theorem, the range of this map has the same dimension as the domain. But if $dim\ V = n$ then by Theorem 5.3 $dim\ V^* = n$ also. Hence the range is *all* of V^*. QED.

Moral 5.7. We know that there are many inner products on any finite dimensional vector space. But given a particular inner product on a real finite dimensional vector space V, the preceding theorem provides a natural way to identify V with V^* *without making any ad hoc choices of basis*. Here is a consequence of this identification that we live with every day.

Derivative versus gradient. Suppose that V is a finite dimensional real vector space and f is a real valued function on V. For any point x in V and any vector $v \in V$ define

$$\partial_v f(x) = \frac{df(x+tv)}{dt}|_{t=0}$$

This is the derivative of f in the direction v. For example if we choose any basis $e_1, ..., e_n$ of V we may write $x = \sum_{j=1}^n x_j e_j$ and then f is just a function of n real variables, $x_1, ..., x_n$. The chain rule then gives

$$\partial_v f(x) = \sum_{j=1}^n v_j (\partial f / \partial x_j)(x)$$

where of course $v = \sum_{j=1}^n v_j e_j$. This sum is clearly linear in v. In other words the map $v \to \partial_v f(x)$ is, for each x, a linear functional on V. One often writes $f'(x)$ for this linear functional. That is, $f'(x)v = \partial_v f(x)$. So $f'(x)$ is in V^* for each $x \in V$. Therefore the *derivative* , f', is a function from V into V^*. If there is no natural way to identify V^* with V then this map to V^* is the only object around that captures the notion of derivative that you're familiar with. But if V has a given inner product, $(\cdot, \cdot)$, then we can identify the linear functional $f'(x)$ (for each x) with an element of V. This is the gradient of f. That is,

$$\nabla f(x) = f'(x)$$

identified to an element of V by Theorem 5.6. Thus

$$f'(x)v = \partial_v f(x) = (\nabla f(x), v).$$

3.2 Problems on Linear Functionals

Definition. A linear functional on a real or complex vector space V is a scalar valued function f on V such that

i) $f(\alpha x) = \alpha f(x)$ for all scalars α and all $x \in V$.
and ii) $f(x + y) = f(x) + f(y)$ for all x and y in V.

Which of the following expressions define linear functionals on the given vector space?

1. $V = R^3$, $x = (x_1, x_2, x_3)$. Explain why *not* if you think not.

a) $f(x) = x_1 + 5x_2$

b) $f(x) = x_1 + 4$

c) $f(x) = x_1^2 + 5x_3$

d) $f(x) = 7$

e) $f(x) = 0$

f) $f(x) = \sin x_2$

g) $f(x) = x_1 x_2$

h) $f(x) = x \cdot u$ where u is a fixed vector and $x \cdot u = \sum_{j=1}^{3} x_j u_j$

2. $V = C([0, 1])$ (real valued continuous functions on $[0, 1]$).

a) $F(\varphi) = \int_0^1 \varphi(t)dt$ for $\varphi \in C([0, 1])$

b) $F(\varphi) = \varphi(3/5)$

c) $F(\varphi) = \varphi(0)$

d) $F(\varphi) = \varphi(0)^2$

e) $F(\varphi) = \varphi(0)\varphi(1)$

f) $F(\varphi) = \int_0^1 \varphi(t) \sin t dt$

g) $F(\varphi) = \int_0^1 \varphi(t)^2 dt$

h) $F(\varphi) = \int_0^1 \varphi(t)^2 t dt$

i) $F(\varphi) = \int_0^1 (\varphi(t)/\sqrt{t}) dt$

j) $F(\varphi) = \int_0^1 (\varphi(t)/t) dt$

k) $F(\varphi) = \int_0^1 (\sin \varphi(t)) dt$

Explain why *not* if you think not.

3. Let V be the vector space of all finitely non–zero real sequences. [A sequence $x = (x_1, x_2, \dots)$ is called finitely non–zero if $\exists N \ni x_k = 0$ for all $k \geq N$.] If $a = (a_1, a_2, a_3, \dots)$ is an arbitrary sequence of real numbers let

$$f_a(x) = \sum_{j=1}^{\infty} a_j x_j \quad \text{for} \quad x \quad \text{in} \quad V. \tag{3.3}$$ 1

a) Show that the series converges for each x in V.

b) Show that $f_a(x)$ is a linear function of x.

c) Show that *every* linear functional on V has the form (1). That is, show that if f is a linear functional on V then there exists a sequence, a, such that $f(x) = f_a(x)\ \forall\ x \in V$.

d) Show that the sequence a, in part c) is unique.

4. Denote by $\mathcal{P}_2$ the space of real valued polynomials of degree less or equal to 2. This is a real vector space of dimension three. (Right?) Define an inner product on $\mathcal{P}_2$ by

$$(p, q) = \int_{-1}^{1} p(t) q(t) dt.$$

You have already admitted that the function on $\mathcal{P}_2$ defined by $L(p) = \int_0^2 p(t) \sin t dt$ is a linear functional. But we also know that every linear functional is given uniquely by an element of the space $\mathcal{P}_2$ with the help of the inner product. Thus there is a unique polynomial f of degree at most 2 such that

$$L(p) = (p, f) \quad \text{for all} \quad p \in \mathcal{P}_2.$$

Find f.

3.3 Generalized functions

If you want to measure the electric field near some point in space you could put a small charged piece of cork there and measure the force exerted by the field on the cork. In this way you have converted the problem of making an electrical measurement to that of making a mechanical measurement. Of course the force on the cork is a (constant multiple of) an average of the forces at each point of the cork. If $\mathbf{E}(x)$ is the field strength at x and ρ is the charge distribution on the cork then the net force on the cork is $\int_{\mathbb{R}^3} \mathbf{E}(x)\rho(x)d^3x$, in appropriate units. In practice (and even in some theories) it is only these averages that you can measure. For example if you really want to measure the field at a point x by the preceding method you would have to place a point charge at x. But classical theory shows that the total electric energy of the field produced by a point charge is infinite. The notion of a point charge is therefore at best an idealization. Of course if you know in advance that the electric field is continuous then you can get better and better approximations to the value of $\mathbf{E}(x)$ by using a sequence of smaller and smaller corks. In the classical theory of electomagnetic fields the electric field tends to be continuous and therefore it makes sense to talk about its value at a point. The quantum theory of eletromagnetic fields, however, recognizes that there are tremendous fluctuations in the field at very small scales and only the averaged field has a meaning.

The need for talking only about averages shows up also in measurement of temperature. A thermometer is clearly measuring the average temperature over the volume of the little bulb at the bottom. If the temerature varies from point to point and is a continuous function of position then you can, in principle, measure the temperature at a point by using a sequence of smaller and smaller bulbs. Recall however that a typical small bulb will contain on the order of 10^{22} molecules. In accordance with statistical mechanics the temperature at a point of a system has a really questionable meaning because temperature is a measure of average kinetic energy of a large bunch of molecules. So at an atomic size of scale temperature at a point is meaningless.

A pairing such as $\int_{\mathbb{R}^3} \mathbf{E}(x)\rho(x)d^3x$, between an "extensive" quantity such as charge ("extensive" means that in a larger volume you have more of the stuff, such as charge, mass, etc.) and an "intensive" quantity, such as the electric field (in a larger volume you don't have more field) occurs often in physics. The integral is linear in ρ and defines a linear functional on the vector space of test charges. The integral is also linear in $\mathbf{E}$ for fixed ρ. We say that the integral is a bilinear pairing between the extensive quantity ρ and the intensive quantity $\mathbf{E}$. Such bilinear pairings between two vector spaces is common. Usually the elements in these dual vector spaces have a physical interpretation, extensive for one and intensive for the other. (Julian Schwinger, in his book "Sources and Fields" emphasizes the duality between sources and fields.)

We are going to develop and use this notion of duality between very smooth functions (test functions) and very "rough" functions (e.g. delta functions.) It has proven to be a great simplifying machinery for understanding partial differential equations as well as Fourier transforms. We are going to apply it to both.

The first step is to understand really smooth functions.

Test functions

Lemma 1. Let

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Then f is infinitely differentiable on the entire real line.

Proof: First, recall that e^t grows faster than any polynomial as $t \to +\infty$. That is, $\lim_{t\to+\infty} p(t)e^{-t} = 0$ for any polynomial. Second, you can see by induction that for $x > 0$ the nth derivative $d^n e^{-1/x} dx^n = p_n(1/x)e^{-1/x}$ for some polynomial p_n. [Convince yourself with the cases $n = 0, 1, 2$.] Third, you can see easily that all of the derivatives of f exist at any point other than $x = 0$, and is zero to the left of 0. The only question then is whats happening at $x = 0$? Sadly, one must go back to the definition of derivative to answer this. But its not so hard. If we know that the first n derivatives exist at $x = 0$ and are zero there then the first and second comments above show that the $n + 1$st *right hand* derivative is

$$\lim_{h\downarrow 0} \frac{p_n(1/h)e^{-1/h} - 0}{h - 0} = 0$$

Of course the left hand derivative is clearly zero. So the two sided derivative exists and is zero. This is the basis for an induction proof. Carry out the case $n = 0$ yourself. QED.

Lemma 2 Let

$$\phi(x) = f(x)f(1 - x)$$

where f is the function constructed in Lemma 1. Then ϕ is an infinitely differentiable function on $\mathbb{R}$ with support contained in the interval $[0, 1]$.

Proof:. ϕ is clearly infinitely differentiable by the repeated application of the product rule for derivatives. To see that ϕ is zero outside the interval $[0, 1]$ draw a picture. QED

Notation. C_c^∞ is the standard notation for the set of infintiely differentiable functions with support in a finite interval. One says that these functions have *compact support*.

We have now constructed one (not identically zero) function in C_c^∞. From this function its easy to construct lots more. For example the function $\psi(x) = 3\phi(2x+7) + 5\phi(4x)^3$ is also in C_c^∞. By scaling and translating the argument of ϕ and taking powers one clearly gets lots of such functions. In FACT there are so many such functions that they are dense in $L^2(\mathbb{R})$. [Remember the concept of density form the chapter on Hilbert space?] One of the homework problems sketches how to show that any *continuous* function with compact support is a limit of functions in C_c^∞.

Notation The space C_c^∞ arises so often that it customarily is given a special notation:

$$\mathcal{D} \equiv C_c^\infty(\mathbb{R}).$$

The dual space of $\mathcal{D}$ is denoted, as usual, $\mathcal{D}^*$.

Terminology An element in $\mathcal{D}^*$ is called a *distribution* or a *generalized function* (according to taste).

Examples 1. Suppose that f is a continuous function on $\mathbb{R}$. Define

$$L_f(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx \text{ for } \phi \in \mathcal{D}. \tag{3.4}$$ 5.1

Then L_f is a linear functional on $\mathcal{D}$. Notice that even if f increases near ∞ (e.g. $f(x) = e^{x^2}$) the integral makes sense because its really an integral over some (and in fact any) interval that supports ϕ. So $L_f \in \mathcal{D}^*$.

2. Let

$$L_\delta(\phi) = \phi(0).$$

Then L_δ is also a linear functional on $\mathcal{D}$. (Clear?) So $L_\delta \in \mathcal{D}^*$. But this example is substantially different from the first one. There is no continuous function f, or even discontinuous function f, such that $L_\delta = \int_{-\infty}^{\infty} f(x)\phi(x)dx$.

3. Its still OK if the function f in Example 1 is not continuous but has only some mild singularities. For example if $f(x) = 1/|x|^{1/2}$ in (3.4) then the integral still exists for any test function ϕ. All we need of f is that $\int_a^b |f(x)|dx < \infty$ for any finite interval $[a, b]$. In particular the function $f(x) = 1/|x|$ won't work in (3.4). The integral doesn't make sense for an arbitrary test function ϕ. One says that f is *locally integrable* if $\int_a^b |f(x)|dx < \infty$ for any finite interval $[a, b]$.

The lesson to be drawn from these examples is this: any continuous function f on R produces a "generalized function" L_f, i.e. an element of $\mathcal{D}^*$ by means of the formula (3.4). But not every element of $\mathcal{D}^*$ comes from a continuous function in this way (or even from a discontinuous function), as we see in Example 2. That's why we call the elements of $\mathcal{D}^*$ generalized functions. Neat terminology, huh?

4. There is an important instance that violates the wisdom of Example 3. Its based strongly on *cancellation of singularity*.

Lemma Let $\phi \in \mathcal{D}$. Then

$$P(\frac{1}{x})(\phi) \equiv \lim_{a \downarrow 0} \int_{|x|>a} \frac{\phi(x)}{x} dx \tag{3.5}$$ [5.3]

exists and is equal to

$$\int_{|x|>1} \frac{\phi(x)}{x} dx + \int_{-1}^{1} \frac{\phi(x) - \phi(0)}{x} dx \tag{3.6}$$ [5.4]

Proof: If $0 < a < b$ then

$$\int_{|x|>a} \frac{\phi(x)}{x} dx = \int_{|x|>b} \frac{\phi(x)}{x} dx + \int_{a<|x|\le b} \frac{\phi(x) - \phi(0)}{x} dx \tag{3.7}$$ [5.5]

because $\int_{a<|x|\le b} \frac{1}{x} dx = 0$. The integrand in the last term in (3.7) is bounded on $\mathbb{R}$ by the mean value theorem. So we can let $a \downarrow 0$ and get a limit. This also proves the validity of the representation (3.6) of this limit. QED

The generalized function $P(\frac{1}{x})$ is called the *Principal Part of* $1/x$. It arises in many contexts. We will see it coming up later in the Feynman propagator.

3.4 Derivatives of generalized functions

Definition Let $T \in \mathcal{D}^*$. The *derivative* of T is the element T' of $\mathcal{D}^*$ given by

$$T'(\phi) = -T(\phi') \quad \text{for all} \quad \phi \in \mathcal{D}. \tag{3.8}$$ D1

What does this definition have to do with our well known notion of derivative of a function? First lets observe that at least T' is indeed a well defined linear functional on $\mathcal{D}$. The reason is that for any $\phi \in \mathcal{D}$ the function ϕ' is again in $\mathcal{D}$ so the right side of (3.8) makes sense. The linearity of T' is clear. Right? To understand why this definition is justified consider the example $T = L_f$ spelled out in (3.4). Suppose that f is actually differentiable in the classical sense. (i.e. in the sense that you grew up with.) Then

$$\begin{aligned}
L_{f'}(\phi) &= \int_{-\infty}^{\infty} f'(x)\phi(x)dx \\
&= \int_a^b f'(x)\phi(x)dx \\
&\quad \text{where } a \text{ and } b \text{ are chosen so that } \phi = 0 \text{ off } (a,b) \\
&= f(x)\phi(x)|_a^b - \int_a^b f(x)\phi'(x)dx \\
&= -\int_a^b f(x)\phi'(x)dx \\
&= -\int_{-\infty}^{\infty} f(x)\phi'(x)dx \\
&= -L_f(\phi')
\end{aligned}$$

So

$$L_{f'}(\phi) = -L_f(\phi'). \tag{3.9}$$ D3

Stare at (3.8) and (3.9). Do you see now why (3.8) is a justifiable definition of derivative of a generalized function T? That's right. It agrees with the classical notion of derivative when $T = L_f$ and f is itself differentiable!!! But (3.8) has a well defined meaning even when its not of the form L_f for some differentiable function f. Lets see what the definition (3.8) gives when $T = L_H$ and H is the non-differentiable function given by

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{3.10}$$

In this case we have, USING (3.8),

$$
\begin{aligned}
T'(\phi) &= -T(\phi') && (3.11)\\
&= -L_H(\phi') && (3.12)\\
&= -\int_0^\infty \phi'(x)dx && (3.13)\\
&= \phi(0) && (3.14)
\end{aligned}
$$

by the fundamental theorem of calculus. So we have

$$(L_H)' = L_\delta. \tag{3.15}$$

D5

You could say (flippantly) that we have now differentiated a non-differentiable function, H and found $H' = \delta$. In truth the perfectly meaningful equation (3.15) is often written as $H' = \delta$. But what does this equation mean? It means (3.15). For the next two weeks we will regard it as immoral to write the equation $H' = \delta$.

Example: $(L_\delta)'(\phi) = -\phi'(0)$

We now have a notion of derivative of a generalized function. Lets end with one more definition.

Definition A sequence T_n of generalized functions converges to a generalized function T if the sequence of numbers $T_n(\phi)$ converges to $T(\phi)$ for each $\phi \in \mathcal{D}$.

3.5 Problems on derivatives and convergence of distributions.

1. Define

$$L(\phi) = \int_{-\infty}^{\infty} |x|\phi(x)dx \quad \text{for } \phi \in C_c^\infty(R).$$

Using the definition

$$T'(\phi) = -T(\phi'),$$

compute the first four derivatives of L; that is, compute $T', ..., T^{(4)}$.

Hint #1. Use the definition of derivative of a distribution.

Hint #2. Use the definition four times to compute the four derivatives.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous (but not necessarily differentiable.) Let

$$u(x, t) = f(x - ct).$$

Show that u is a solution to the wave equation

$$\partial^2 u/\partial t^2 = c^2 \partial^2 u/\partial x^2$$

in the distribution sense (sometimes called the weak sense.)

Nota Bene: Since f is not necessarily differentiable you cannot use f' in the classical sense.

3. Does $\sum_{n=1}^{\infty} \delta(x - n)$ converge in the distribution sense?

4. Let $p_t(x) = (2\pi t)^{-1/2} e^{-x^2/(2t)}$. Find the folowing limits if they exist.

a. the pointwise limit as $t \downarrow 0$.

b. the $L^2(\mathbb{R})$ limit.

c. the limit in $\mathcal{D}^*$.

You may use the following FACT that will be proved later

$$\int_{-\infty}^{\infty} p_t(x)dx = 1 \quad \forall t > 0.$$

5. Let ϕ be the function constructed in Lemma 2, except multiply it by a positive constant such that

$$\int_{\mathbb{R}} \phi(x)dx = 1$$

Such a constant can be found because the original ϕ is nonnegative and has a strictly positive integral. Then let

$$\phi_n(x) = n\phi(nx)$$

Our goal is to show that ϕ_n converges in some sense to a delta function.

a. Show that $\int_{\mathbb{R}} \phi_n(x)dx = 1$ for all positive integers n.

b. Suppose that g is a continuous function on $\mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi_n(x)g(x)dx = g(0).$$

c. Use part b. and the definition of convergence of distributions to show that

$$L_{\phi_n} \text{ converges in the weak sense to } L_\delta.$$

6. Prove that, for all $\phi \in \mathcal{D}$,

$$\lim_{\epsilon\downarrow 0} \int_{-\infty}^{\infty} \frac{\phi(x)}{x+i\epsilon}dx = P(\frac{1}{x})(\phi) - i\pi\phi(0) \qquad (3.16) \quad \boxed{\texttt{5.8}}$$

This is often written as

$$\lim_{\epsilon\downarrow 0} \frac{1}{x+i\epsilon} = P(\frac{1}{x}) - i\pi\delta \quad \text{weak sense} \qquad (3.17) \quad \boxed{\texttt{5.9}}$$

Hint: Review the proof of the lemma at the end of Section 3.3.

3.6 Distributions over $\mathbb{R}^n$

The extension of the one dimensional notion of distribution to an n dimensional distribution is straightforward. Denote by $\mathcal{D}$ the vector space of infinitely differentiable functions on $\mathbb{R}^n$ which are zero outside of a large cube (that can depend on the function.) This space is sometimes denoted $C_c^\infty(\mathbb{R}^n)$. There are plenty of such functions. For example if $\phi_1, \ldots, \phi_n$ are each in $C_c^\infty(\mathbb{R})$ then the function $\psi(x_1, \ldots, x_n) = \phi(x_1) \cdots \phi(x_n)$ is in $C_c^\infty(\mathbb{R}^n)$. (What cube could you use?) So is any finite linear combination of these products. A distribution over $\mathbb{R}^n$ is defined as a linear functional on $\mathcal{D}$. Of course we can make up examples of such n-dimensional distributions similar to the ones we already know in one dimension: Let

$$L_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x) d^n x \tag{3.18}$$

As long as $|f|$ has a finite integral over every cube this expression makes sense and defines a linear functional on $\mathcal{D}$, just as in one dimension. We also have the n-dimensional δ "function" defined by

$$L_\delta(\phi) = \phi(0). \tag{3.19}$$

The difference from one dimension shows up when we consider differentiation. We now have partial derivatives. Here is the definition of partial derivative (as if you couldn't guess).

$$\frac{\partial T}{\partial x_k}(\phi) = -T(\partial\phi/\partial x_k). \tag{3.20}$$

When $n = 3$ every distribution has a very intuitive interpretation as an arrangement of charges, dipoles, quadrupoles, etc. I'm going to explain this interpretation in class in more detail. It gives physical meaning to every element of $\mathcal{D}^*$!!!

3.7 Poisson's Equation

We will write as usual $r = |x|$ in $\mathbb{R}^3$.

Theorem 3.1

$$\Delta \frac{1}{r} = -4\pi\delta. \tag{3.21}$$ P1

in the distribution sense. That is,

$$\Delta L_{1/r} = -4\pi L_\delta$$

We will break the proof up into several small steps.

Lemma 3.2 *At* $r \neq 0$

$$\Delta(1/r) = 0$$

Proof. $\partial(1/r)/\partial x = -x/r^3$ and $\partial^2(1/r)/\partial x^2 = -1/r^3 + 3\frac{x^2}{r^5}$. So

$$\Delta(1/r) = -3/r^3 + 3\frac{x^2+y^2+z^2}{r^5} = -3/r^3 + 3/r^3 = 0.$$

QED.

In view of this lemma you can see that we have only to deal now with the singularity at $r = 0$. Our notion of weak derivative is just right for doing this.

The trick is to avoid the singularity until after one does some clever integration by parts (in the form of the divergence theorem). In case you forgot your vector calculus identities a self contained review is at the end of this section. I want to warn you that this is not the kind of proof that you are likely to invent yourself. But the techniques are so frequently occurring that there is some virtue in following it through at least once in one's life.

Lemma 3.3 *Let* $\phi \in \mathcal{D}$. *Then*

$$\int_{\mathbb{R}^3} (1/r)\Delta\phi(x)dx = \lim_{\epsilon \to 0} \int_{r \geq \epsilon} (1/r)\Delta\phi dx$$

Proof: The difference between the left and the right sides before taking the limit is at most (use spherical coordinates in the next step)

$$|\int_{r\leq\epsilon}(1/r)\Delta\phi d^3x| \leq \max_{x\in\mathbb{R}^3}|\Delta\phi(x)|\int_{r\leq\epsilon}(1/r)d^3x = \max_{x\in\mathbb{R}^3}|\Delta\phi(x)|2\pi\epsilon^2 \to 0$$

QED.

Before really getting down to business lets apply the definitions.

$$\Delta T_{1/r}(\phi) = \sum_{j=1}^{3}(\partial^2/\partial x_j^2)T_{1/r}(\phi) \tag{3.22}$$

$$= -\sum_{j=1}^{3}(\partial/\partial x_j)T_{1/r}(\partial\phi/\partial x_j) \tag{3.23}$$

$$= T_{1/r}(\Delta\phi) \tag{3.24}$$

$$= \int_{\mathbb{R}^3}(1/r)\Delta\phi(x)d^3x \tag{3.25}$$

$$= \lim_{\epsilon\to 0}\int_{r\geq\epsilon}(1/r)\Delta\phi(x)d^3x. \tag{3.26}$$

So what we really need to do is show that this limit is $-4\pi\phi(0)$. To this end we are going to apply some standard integration by parts identities in the "OK" region $r \geq \epsilon$.

$$C_\epsilon := \int_{r\geq\epsilon}(1/r)\Delta\phi(x)d^3x \tag{3.27}$$

$$= \int_{r\geq\epsilon}\nabla\cdot\big(\frac{1}{r}\nabla\phi - \phi\nabla(\frac{1}{r})\big)d^3x \quad \text{by identity (3.39)} \tag{3.28}$$

$$= \int_{r=\epsilon}(\frac{1}{r}\nabla\phi\cdot\mathbf{n} - \phi(\nabla\frac{1}{r})\cdot\ \mathbf{n})dA \quad \text{by the divergence theorem} \tag{3.29}$$

where $\mathbf{n}$ is the unit normal pointing toward the origin. The other boundary term in this integration by parts identity is zero because we can take it over a sphere so large that ϕ is zero on and outside it.

Now

$$|\int_{r=\epsilon}(1/r)(\nabla\phi\cdot\mathbf{n})dA| = \frac{1}{\epsilon}|\int_{r=\epsilon}(\nabla\phi\cdot\mathbf{n}dA| \tag{3.30}$$

$$\leq \frac{1}{\epsilon}(\max|\nabla\phi|)4\pi\epsilon^2 \tag{3.31}$$

$$\to 0 \tag{3.32}$$

as $\epsilon \downarrow 0$. This gets rid of one of the terms in C_ϵ in the limit. For the other one just note that $(\nabla\frac{1}{r})\cdot\mathbf{n} = -\partial(1/r)/\partial r = 1/r^2$. So

$$-\int_{r=\epsilon}\phi(\nabla\frac{1}{r})\cdot\mathbf{n})dA = -\frac{1}{\epsilon^2}\int_{r=\epsilon}\phi(x)dA \tag{3.33}$$

$$= -\frac{1}{\epsilon^2}\int_{r=\epsilon}\phi(0)dA - \frac{1}{\epsilon^2}\int_{r=\epsilon}(\phi(x)-\phi(0))dA \tag{3.34}$$

$$= -4\pi\phi(0) - \frac{1}{\epsilon^2}\int_{r=\epsilon}(\phi(x)-\phi(0))dA \tag{3.35}$$

Only one more term to get rid of!

$$\frac{1}{\epsilon^2}|\int_{r=\epsilon}(\phi(x)-\phi(0))dA| \leq \max_{|x|=\epsilon}|\phi(x)-\phi(0)|\cdot 4\pi \to 0$$

because ϕ is continuous at $x = 0$. This proves (3.21).

Vector calculus identities.

If f is a real valued function and G is a vector field, both defined on some region in $\mathbb{R}^3$ then

$$\nabla\cdot(fG) = (\nabla f)\cdot G + f\nabla\cdot G \tag{3.36}$$ P5

Application #1. Take $f = 1/r$ and $G = \nabla\phi$. Then we get

$$\nabla\cdot(\frac{1}{r}\nabla\phi) = (\nabla\frac{1}{r})\cdot\nabla\phi + \frac{1}{r}\Delta\phi \text{ wherever } r \neq 0. \tag{3.37}$$ P6

Application #2. Take $f = \phi$ and $G = \nabla\frac{1}{r}$. Then we get

$$\nabla\cdot(\phi\nabla\frac{1}{r}) = (\nabla\phi)\cdot(\nabla\frac{1}{r}) + \phi\Delta\frac{1}{r} \text{ wherever } r \neq 0 \tag{3.38}$$ P7

But $\Delta\frac{1}{r} = 0$ wherever $r \neq 0$. So subtracting (3.38) from (3.37) we find

$$\frac{1}{r}\Delta\phi = \nabla\cdot(\frac{1}{r}\nabla\phi - \phi\nabla\frac{1}{r}) \text{ wherever } r \neq 0. \tag{3.39}$$ P8

This is the identity we need in the proof of (3.21).

4 The Fourier Transform

The Fourier transform of a complex valued function on the line is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} f(x) dx \tag{4.1}$$ F1

Here ξ runs over $\mathbb{R}$. The most useful aspect of this transform of functions is that it interchanges differentiation and multiplication. Thus if you differentiate under the integral sign you get

$$\frac{d}{d\xi} \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} \{ixf(x)\} dx. \tag{4.2}$$ F2

So

$$(\text{Fourier transform of} \{ixf(x)\})(\xi) = \frac{d}{d\xi} \hat{f}(\xi). \tag{4.3}$$

And an integration by parts (never mind the boundary terms) clearly gives

$$-i\xi \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} f'(x) dx. \tag{4.4}$$ F3

So

$$\widehat{f'}(\xi) = -i\xi \hat{f}(\xi) \tag{4.5}$$

We will see later that these formulas allow one to solve some partial differential equations. Moreover in quantum mechanics these two formulas amount to the statement that the Fourier transform interchanges P and Q (momentum and position operators.)

But the usefulness of these formulas depends crucially on the fact that one can also transform back and recover f from $\hat{f}$. To this end there is an inversion formula that does the job. Our goal is to establish the most useful properties of the Fourier transform and in particular to derive the inversion formula and show how to use it to solve PDEs.

To begin with we must understand how to give honest meaning to the formula (4.1). Since the integral is over an infinite interval there is a convergence question right away. Suppose that

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty. \tag{4.6}$$ F4

We will write $f \in L^1$ if (4.6) holds. If $f \in L^1$ then there is no problem with the existence of the integral in (4.1) because $\lim_{a\to\infty} \int_{-a}^{a} e^{i\xi x} f(x)dx$ exists. [Proof: $|\big(\int_{-a}^{a} - \int_{-b}^{b}\big) e^{i\xi x} f(x)dx| \le \int_{a\le |x| \le b} |f(x)|dx \to 0$ as $a \le b \to \infty$. Use the Cauchy convergence criterion now.]

Of course even if $f \in L^1$ it can happen that f' is not in L^1 and/or that $xf(x)$ is not in L^1. This is a nuisance in dealing with the identities (4.2) and (4.4). We are going to restrict our attention for a while to a class of functions that will make these issues easy to deal with.

Definition. A function f on $\mathbb{R}$ is said to be *rapidly decreasing* if

$$|x^n f(x)| \le M_n, \quad n = 0, 1, 2, ... \tag{4.7}$$

for some real numbers M_n. In words: $x^n f(x)$ is bounded on $\mathbb{R}$ for each n.

Examples 1. e^{-x^2} is rapidly decreasing.

2. $\frac{1}{x^2+1}$ is not rapidly decreasing because (4.7) only holds for $n = 0, 1, 2$ but not for $n = 3$ or more.

3. If f is rapidly decreasing then so is $x^5 f(x)$ because $x^n x^5 f(x) = x^{n+5} f(x)$ which is bounded in accordance with (4.7). Just replace n by $n+5$ in (4.7). Since any finite linear combination of rapidly decreasing functions is also rapidly decreasing we see that $p(x)f(x)$ is rapidly decreasing for any polynomial p if f is rapidly decreasing.

4. So by examples 1. and 3. we see that $p(x)e^{-x^2}$ is rapidly decreasing for any polynomial p.

5. Any function in $C_c^\infty(\mathbb{R})$ is rapidly decreasing.

6. Summary: The space of rapidly decreasing functions is a vector space and is closed under multiplication by any polynomial. And besides, there are lots of these functions.

Lemma Any rapidly decreasing function is in L^1.

Proof: Apply (4.7) for $n = 0$ and $n = 2$ to conclude that

$$|(1 + x^2)f(x)| \le M$$

for some real number M. So

$$\int_{-\infty}^{\infty} |f(x)|dx \le M \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx < \infty.$$

QED

Using only rapidly decreasing functions in (4.2) will allow us not to have to worry about whether f *and* $xf(x)$ are both in L^1. Neat, huh?

But to use (4.4) we still need to worry about whether f' is in L^1. So we are going to restrict our attention for a while, even further, to a class of functions that makes both (4.2) and (4.4) easy to deal with.

Notation: $\mathcal{S}$ will denote the set of C^∞ functions on $\mathbb{R}$ such that f and each of its derivatives is rapidly decreasing.

Examples

7. e^{-x^2} is infinitely differentiable and each derivative is just a polynomial times e^{-x^2}. We've already seen that these functions are rapidly decreasing. So the function e^{-x^2} is in $\mathcal{S}$.

8. $\frac{1}{x^2+1}$ is infinitely differentiable but is not rapidly decreasing. So this function is not in $\mathcal{S}$.

9. Now here is a real nice thing about this space $\mathcal{S}$. If $f \in \mathcal{S}$ then any polynomial, p, times any derivative of f is again in $\mathcal{S}$. CHECK THIS against the definitions! I know this may seem too good to believe. But we do know that there are lots of functions in $\mathcal{S}$. All of C_c^∞ is contained in $\mathcal{S}$. And besides there are more, as we saw in Example 7.

STATUS: If $f \in \mathcal{S}$ then f and all of its derivatives are in L^1. So both formulas (4.2) and (4.4) make sense. Moreover they are both correct because the boundary terms that we ignored in deriving (4.4) are indeed zero, since these functions go to zero so quickly at ∞. (Check this at your leisure!)

In truth, here is the reason that the space $\mathcal{S}$ is so great.

Invariance Theorem. If $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$.

Proof: First notice that for any function f in L^1 we have the bound

$$|\hat{f}(\xi)| = |\int_{-\infty}^{\infty} e^{i\xi x} f(x)dx| \tag{4.8}$$

$$\leq \int_{-\infty}^{\infty} |f(x)|dx \tag{4.9}$$

$$= \|f\|_1. \tag{4.10}$$

Since the right side doesn't depend on ξ $\hat{f}$ is bounded.

Now suppose that $f \in \mathcal{S}$. Then so is f', f'', etc. So f', f'' etc. are all in L^1. By (4.4) it now follows that $\xi^n \hat{f}(\xi)$ is bounded for each $n = 0, 1, 2....$ So $\hat{f}$ is rapidly decreasing. But $d\hat{f}(\xi)/d\xi$ is the Fourier transform of $ixf(x)$ which we have seen is also in $\mathcal{S}$. So $d\hat{f}(\xi)/d\xi$ is also rapidly decreasing. And so on. QED.

Here is what the inversion formula will look like

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iy\xi}\hat{f}(\xi)d\xi. \tag{4.11}$$ F8

Since we already know that $\hat{f}$ is in $\mathcal{S}$ we know that the right hand side of (4.11) makes sense. Contrast this with the example that you worked out in the homework: Take $f(x) = 1$ if $|x| \leq 1$ and $f = 0$ otherwise. Then $f \in L^1$ so $\hat{f}$ makes sense. But $\hat{f}$ itself decreases so slowly at ∞ that its not in L^1. So (4.11) doesn't make sense. Aren't you glad that we are focusing on functions in $\mathcal{S}$?

Notation:

$$\check{g}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iy\xi}g(\xi)d\xi \tag{4.12}$$ F9

Inversion Theorem. For $f \in \mathcal{S}$ equation (4.11) holds.

In other words : $\check{\hat{f}} = f$.

We are going to spend the next few pages proving this formula. (Not because I fear that you might not trust me, but because the proof derives some very useful identities along the way.)

4.1 The Fourier Inversion formula on $\mathcal{S}(R)$.

To begin, here are some explicit computations of some important Fourier transforms.

Gaussian identities

Let

$$p_t(x) = \frac{1}{\sqrt{2\pi t}}\ e^{-x^2/2t} \tag{4.13}$$ F10

Then we have the following three identities.

$$\int_{-\infty}^{\infty} p_t(x)dx = 1 \tag{4.14}$$ F11

$$\hat{p}_t(\xi) = e^{-t\xi^2/2} \tag{4.15}$$ F12

$$(e^{-\frac{t}{2}\xi^2})^{\check{}}(x) = \frac{1}{\sqrt{2\pi t}}\ e^{-x^2/2t} = p_t(x). \tag{4.16}$$ F13

Proof of (F11 4.14) [Sneaky use of polar coordinates.]

$$\{\int_{-\infty}^{\infty} p_t(x)dx\}^2 = \int_{\mathbb{R}^2} p_t(x)p_t(y)dxdy \tag{4.17}$$

$$= \frac{1}{2\pi t}\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2t}}dxdy \tag{4.18}$$

$$= \frac{1}{2\pi t}\int_0^{\infty}\int_0^{2\pi} e^{-r^2/2t}rdrd\theta \tag{4.19}$$

$$= 1 \tag{4.20}$$

(Use the substitution $s = r^2/(2t)$ in the last step.)

Proof of (F12 4.15)

To compute $\hat{p}_t(\xi)$ we need first to multiply $p_t(x)$ by $e^{ix\xi}$ before integration. The exponent is a quadratic function of x for which we can complete the square thus:

$$-\frac{x^2}{2t} + ix\xi = -\frac{1}{2t}(x - it\xi)^2 - t\xi^2/2$$

Hence

$$\hat{p}_t(\xi) = e^{-t\xi^2/2}\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-\frac{1}{2t}(x-it\xi)^2}dx \tag{4.21}$$ F16

The coefficient of $e^{-t\xi^2/2}$ looks "just like" $\int_{-\infty}^{\infty} p_t(x)dx$ which is one. This would prove (4.15). But not so fast. x has been translated by the imaginary number $it\xi$. You can't just translate the argument to get rid of it. Here is how to get rid of it. Let $a = -t\xi$. and suppose $a > 0$ just so that I can draw pictures verbally. Take a large positive number R and consider the rectangular contour in the complex plane which starts at $-R$ on the real axis, moves along the real axis to R and then moves vertically up to the point $R + ia$. From there move horizontally, west to $-R + ia$ and then south back to the point $-R$. Since $e^{-z^2/(2t)}$ is an entire function, the integral of this function around this rectangular contour is zero. Otherwise put, $\int_{-R}^{R} e^{(x+ia)^2/(2t)} = \int_{-R}^{R} e^{-x^2/(2t)}dx$ plus a little contribution from the vertical sides of the rectangle. On the right end the integral is at most a times the maximum value of the integrand on that segment. But $|e^{-(R+iy)^2/(2t)}| = e^{(-R^2+y^2)/(2t)}$, which goes to zero quite quickly for $0 \le y \le a$ as $R \to \infty$. So the integrals along the end segments go to zero as $R \to \infty$. So it is indeed true that the coefficient of $e^{-t\xi^2}$ in (4.21) equals $\int_{-\infty}^{\infty} p_t(x)dx$, which is one. QED

Proof of (4.16) $e^{-t\xi^2/2} = \sqrt{\frac{2\pi}{t}}p_{1/t}(\xi)$ by (4.13). Since $p_{1/t}$ is even we have $\check{(}e^{-t\xi^2/2}) = (1/2\pi)\sqrt{\frac{2\pi}{t}}(\hat{p}_{1/t})(x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$ by (4.15). QED

Definition 4.1 The *convolution* of two functions f and g on R is given by

$$(f * g)(x) = \int_R f(x-y)g(y)dy \tag{4.22}$$ F21

lemF1 **Lemma 4.2** *Let g be a bounded continuous function on $\mathbb{R}$. Then for each x*

$$\lim_{t\downarrow 0}(p_t * g)(x) = g(x).$$

<u>Proof</u>: Since $\int_{-\infty}^{\infty} p_t(x)dx = 1$ we have

$$(p_t * g)(x) - g(x) = (g * p_t)(x) - g(x) \tag{4.23}$$

$$= \int g(x-y)p_t(y)dy - g(x) \tag{4.24}$$

$$= \int_{-\infty}^{\infty} (g(x-y) - g(x))p_t(y)dy. \tag{4.25}$$

Given $\varepsilon > 0$, $\exists\ \delta > 0 \ni |g(x-y) - g(x)| < \varepsilon$ if $|y| < \delta$.

$$\therefore |p_t * g(x) - g(x)| \leq \int_{-\delta}^{\delta} |g(x-y) - g(x)| p_t(y) dy \tag{4.26}$$

$$+ \int_{|y|>\delta} |g(x-y) - g(x)| p_t(y) dy \tag{4.27}$$

$$\leq \varepsilon \int_{-\delta}^{\delta} p_t(y) dy + 2|g|_\infty \int_{|y|>\delta} p_t(y) dy \tag{4.28}$$

$$\leq \varepsilon + 2|g|_\infty \int_{|y|>\delta} p_t(y) dy. \tag{4.29}$$

$$\therefore \overline{\lim_{t \downarrow 0}} |(p_t * g)(x) - g(x)| \leq \varepsilon + 2|g|_\infty \overline{\lim_{t \downarrow 0}} \int_{|y| \geq \delta} p_t(y) dy.$$

But

$$\int_{y \geq \delta} p_t(y) dy = \frac{2}{\sqrt{2\pi t}} \int_\delta^\infty e^{-y^2/2t} dy \leq \frac{2}{\sqrt{2\pi t}} \int_\delta^\infty \frac{y}{\delta} e^{-y^2/2t} dy \tag{4.30}$$

$$= \frac{2}{\delta\sqrt{2\pi t}} [-te^{-y^2/2t}]_\delta^\infty = \frac{2}{\delta\sqrt{2\pi}} \sqrt{t} e^{-\delta^2/2t} \rightarrow \text{ as } t \downarrow 0. \tag{4.31}$$

Therefore

$$\overline{\lim_{t \downarrow 0}} |(p_t * g)(x) - g(x)| \leq \varepsilon\ \forall\ \varepsilon > 0.$$

Hence

$$\overline{\lim_{t \downarrow 0}} |(p_t * g)(x) - g(x)| = 0.$$

Q.E.D.

Lemma 4.3 *If f and g are in $\mathcal{S}(R)$ then*

$$(\check{f} * g)(x) = \frac{1}{(2\pi)^n} \int f(\xi) e^{-i\xi \cdot x} \widehat{g}(\xi).$$

Proof:

$$(\check{f} * g)(x) = \int_R \check{f}(x-y)g(y)dy \tag{4.32}$$

$$= \frac{1}{(2\pi)^n}\int_R\int_R f(\xi)e^{-i\xi\cdot(x-y)}g(y)d\xi dy \tag{4.33}$$

$$= \frac{1}{(2\pi)^n}\int_R\int_R f(\xi)e^{-i\xi\cdot x}e^{i\xi\cdot y}g(y)dyd\xi \tag{4.34}$$

$$= \frac{1}{(2\pi)^n}\int_R f(\xi)e^{-i\xi\cdot x}\widehat{g}(\xi)d\xi. \tag{4.35}$$

Q.E.D.

Theorem 4.4 *If g is in $\mathcal{S}(R)$ then*

$$(\widehat{g})\check{}(x) = g(x).$$

Proof. In the preceding lemma put $f(\xi) = e^{-t\xi^2/2}$. Then, by (4.16), $\check{f}(x) = p_t(x)$. Thus

$$(p_t * g)(x) = \frac{1}{2\pi}\int_R e^{-t\xi^2}e^{-i\xi\cdot x}\widehat{g}(\xi)d\xi.$$

Let $t \downarrow 0$. Use Lemma 4.2 on the left and Dom. Conv. theorem on the right to get

$$g(x) = \frac{1}{2\pi}\int_R e^{-i\xi\cdot x}\widehat{g}(\xi)d\xi.$$

■

The asymmetric way in which the factor 2π occurs in the inversion formula is sometimes a confusing nuisance. It is useful to distribute this factor among the forward and backward transforms. Define

$$(\mathcal{F}g)(\xi) = (2\pi)^{-1/2}\hat{g}(\xi) \tag{4.36}$$

The factor in front of $\hat{g}$ has clearly no effect on the one-to-one or ontoness property of the map $g \to \hat{g}$. That is, $\mathcal{F}$ is a one-to-one map of $\mathcal{S}(R)$ onto $\mathcal{S}(R)$. However the factor makes for a nice identity:

corFP1 **Corollary 4.5** *(Plancherel formula for $\mathcal{S}$) If f and g are in $\mathcal{S}(R)$ then*

$$(\mathcal{F}f, \mathcal{F}g) = (f, g) \tag{4.37}$$

Explicitly, this says

$$\int_R (\mathcal{F}f)(\xi)\overline{(\mathcal{F}g)(\xi)}d\xi = \int_R f(x)\overline{g(x)}dx \tag{4.38}$$

Proof. Phrasing this identity directly in terms of $\hat{f}$ and $\hat{g}$, it asserts that

$$\int_R \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi = (2\pi)\int_R f(x)\overline{g(x)}dx$$

But $\overline{\widehat{g}(\xi)} = \overline{\int g(x)e^{ix\cdot\xi}dx} = \int \overline{g(x)}e^{-ix\cdot\xi}dx$. Hence

$$\int_R \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi = \int_R\int_R \widehat{f}(\xi)\overline{g(x)}e^{-ix\cdot\xi}dxd\xi \tag{4.39}$$

$$= \int_R \Big(\int_R \widehat{f}(\xi)e^{-ix\cdot\xi}\overline{g(x)}d\xi\Big)dx \tag{4.40}$$

$$= \int_R (2\pi)f(x)\overline{g(x)}dx. \tag{4.41}$$

■

Q.E.D.

4.2 The Fourier transform over R^n

The definitions, key formulas, theorems and proofs for the Fourier transform over $\mathbb{R}^n$ are nearly identical to those over $\mathbb{R}$. In this section we are going to summarize the results we've obtained so far and formulate them over $\mathbb{R}^n$.

The Fourier transform of a complex valued function on R^n is

$$\hat{f}(\xi) = \int_{R^n} e^{i\xi\cdot x} f(x)dx \tag{4.42}$$

F51

Here ξ runs over R^n. Just as in one dimension, one must pay some attention to the meaningfulness of this integral. But the ideas are similar.

As in one dimension, the Fourier transform over R^n interchanges multiplication and differentiation. The analog of (4.2) is

$$\frac{\partial}{\partial \xi_j}\hat{f}(\xi) = \int_{R^n} e^{i\xi\cdot x}\{ix_j f(x)\}dx. \tag{4.43}$$

F52

So

$$(\text{Fourier transform of}\{ix_j f(x)\})(\xi) = \frac{\partial}{\partial \xi_j}\hat{f}(\xi). \tag{4.44}$$

An integration by parts (never mind the boundary terms) clearly gives the analog of (4.4):

$$-i\xi_j \hat{f}(\xi) = \int_{R^n} e^{i\xi\cdot x}(\partial f/\partial x_j)(x)dx. \tag{4.45}$$

F53

So

$$\widehat{\partial f/\partial x_j}(\xi) = -i\xi_j \hat{f}(\xi) \tag{4.46}$$

We will see later that these formulas allow one to solve some partial differential equations. Moreover in quantum mechanics these two formulas amount to the statement that the Fourier transform interchanges P_j and Q_j (momentum and position operators.)

We say that a function f is in $L^1(R^n)$ if

$$\int_{R^n} |f(x)|d^n x < \infty \tag{4.47}$$

The formula (4.42) makes perfectly good sense if $f \in L^1(R^n)$. But in the end we are going to give meaning to (4.42) for a much larger class of functions and generalized functions, including e.g. delta functions and their derivatives. To this end we need the n-dimensional analog of $\mathcal{S}$.

Definition 4.6 *A function f on R^n is said to be rapidly decreasing if*

$$|x|^k|f(x)| \leq M_k, \quad k = 0, 1, 2, ... \tag{4.48}$$ F55

for some real numbers M_k. In words: $|x|^k f(x)$ is bounded on R^n for each k.

In Exercise 4 you will have the opportunity to show that the condition (F55 4.48) is equivalent to the statement that for any polynomial $p(x_1, \ldots, x_n)$ in n real variables, the product $p(x)f(x)$ is bounded.

lemF20 **Lemma 4.7** *Any rapidly decreasing function on R^n is in $L^1(R^n)$.*

Definition 4.8 *$\mathcal{S}(R^n)$ is the set of functions f in $C^\infty(R^n)$ such that f and each of its partial derivatives are rapidly decreasing.*

Theorem 4.9 *a. If f is in $\mathcal{S}(R^n)$ then $\hat{f}$ is also in $\mathcal{S}(R^n)$.*
b. The map $f \to \hat{f}$ is a one-to-one linear map of $\mathcal{S}(R^n)$ onto $\mathcal{S}(R^n)$.
c. The inverse is given by the inversion formula

$$f(y) = (2\pi)^{-n} \int_{R^n} e^{-i\xi\cdot y} \hat{f}(\xi) d^n\xi. \tag{4.49}$$ F60

Definition 4.10 The *convolution* of two functions f and g on R^n is given by

$$(f * g)(x) = \int_{R^n} f(x-y)g(y)d^n y \tag{4.50}$$ F61

The important identity in the next theorem is the basis for the application of the Fourier transform to solution of partial differential equations.

Theorem 4.11

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi) \tag{4.51}$$ F65

Proof. The proof consists of the following straight forward computation. A reader who is concerned about the validity of any of these steps should simply assume that f and g are in $\mathcal{S}(R^n)$, although the identity is valid quite a bit more generally.

$$\widehat{f * g}(\xi) = \int_{\mathbb{R}^n} (f * g)(x) e^{ix\cdot\xi} dx \tag{4.52}$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) g(y) dy e^{ix\cdot\xi} dx \tag{4.53}$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big(f(x-y) e^{i(x-y)\cdot\xi}\big) dx g(y) e^{iy\cdot\xi} dy \tag{4.54}$$

$$= \int_{\mathbb{R}^n} \hat{f}(\xi) g(y) e^{iy\cdot\xi} dy \tag{4.55}$$

$$= \hat{f}(\xi)\hat{g}(\xi) \tag{4.56}$$

■

Theorem 4.12 *(Plancherel formula) Define $\mathcal{F}\phi = (2\pi)^{-n/2}\hat{\phi}$. Then*

$$\|\mathcal{F}\phi\|_{L^2(R^n)} = \|\phi\|_{L^2(R^n)} \tag{4.57}$$ F70

This is the Plancherel formula. As a consequence $\mathcal{F}$ is a unitary operator on $L^2(R^n)$

This is a minor restatement of Corollary corFPl 4.5 and extension to R^n.

Finally, here are the n-dimensional analogs of the important Gaussian identities (F11 4.14) - (F13 4.16).

Gaussian identities over R^n.

Let

$$p_t(x) = \frac{1}{\sqrt{(2\pi t)^n}}\; e^{-|x|^2/2t} \quad x \in R^n. \tag{4.58}$$ F80

Then

$$\int_{R^n} p_t(x) dx = 1 \tag{4.59}$$ F81

$$\hat{p}_t(\xi) = e^{-t|\xi|^2/2} \tag{4.60}$$ F82

$$(e^{-\frac{\check{t}}{2}|\xi|^2})(x) = \frac{1}{\sqrt{(2\pi t)^n}}\; e^{-|x|^2/2t} = p_t(x). \tag{4.61}$$ F83

4.3 Tempered Distributions

Definition 4.13 *A tempered distribution on R^n is a linear functional on $\mathcal{S}(R^n)$.*

Example 4.14 $(n = 1)$ Suppose that $f : \mathbb{R} \to \mathbb{C}$ satisfies

$$|f(x)| \le const.(1 + |x|^n) \text{for some } n \ge 0 \tag{4.62}$$

F30

In this case we say that f has *polynomial growth.* E.g. $x^3 \sin x$ has polynomial growth. (Take $n = 3$ in (4.62). Of course $n = 4$ will also do.) If f has polynomial growth then the integral

$$L_f(\phi) \equiv \int_{\mathbb{R}} f(x)\phi(x)dx \tag{4.63}$$

F31

exists when $\phi \in \mathcal{S}$ because $|f(x)\phi(x)| \le const.(1+|x|^n)(1+|x|^{n+2})^{-1}$, which goes to zero at ∞ like x^{-2} and so is integrable. Clearly L_f is linear. Thus any function of polynomial growth determines in this natural way a linear functional on $\mathcal{S}$. Remember that we did not need polynomial growth of f when we made L_f into a linear functional on $\mathcal{D}$. So we have a "smaller" dual space now. But the delta distribution and its derivatives are in this smaller dual space anyway because

$$L_\delta(\phi) \equiv \phi(0)$$

is a meaningful linear functional on $\mathcal{S}$. (And similarly for $\delta^{(k)}$.)

Example 4.15 We can't allow the function $f(x) = e^{2x^2}$ in (4.63) because the function $\phi(x) = e^{-x^2}$ is in $\mathcal{S}$ and so (4.63) wouldn't make sense.

Definition 4.16 The Fourier transform of an element $L \in \mathcal{S}^*$ is defined by

$$\hat{L}(\phi) = L(\hat{\phi}) \quad \text{for } \phi \in \mathcal{S}. \tag{4.64}$$

F33

Now you can see the virtue of using $\mathcal{S}$ as our new test function space: the right hand side of (4.64) makes sense precisely because $\hat{\phi}$ is back in $\mathcal{S}$ when ϕ is in $\mathcal{S}$. After all, L is only defined on $\mathcal{S}$. Had we attempted to use $\mathcal{D}$ this definition would have failed because $\hat{\phi}$ is *never* in $\mathcal{D}$ when $\phi \in \mathcal{D}$ (if $\phi \neq 0$.)

Example 4.17 $\hat{L}_\delta = L_1$. (In the outside world this is usually written $\hat{\delta} = 1$.

Proof. Let $\phi \in \mathcal{S}$. Then

$$\begin{aligned} \hat{L}_\delta(\phi) &= L_\delta(\hat{\phi}) \text{ by def. (4.64)} && (4.65)\\ &= \hat{\phi}(0) \text{ by the def. of } L_\delta && (4.66)\\ &= \int_{\mathbb{R}} \phi(x)dx \text{ by the def. of Fourier trans. at 0} && (4.67)\\ &= L_1(\phi) \;\; \text{by yet another definition} && (4.68) \end{aligned}$$

■

One thing to take away from this proof is that every step is just the application of some definition. There is no mysterious computation. Let this be a lesson to all of us!

Example 4.18

$$\hat{L}_1 = 2\pi L_\delta \tag{4.69}$$ F40

After you have achieved a suitable state of sophistication you can write this as

$$\hat{1} = 2\pi\delta.$$

Or even as!!

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixy}dx = \delta(y)$$

(if you don't feel too uncomfortable with the seeming nonsense of the left side. But you better get used to it. Thats how (4.69) is written in the outside world.)

Proof. of (4.69):

$$\begin{aligned} \hat{L}_1(\phi) &= L_1(\hat{\phi}) \text{ by def. (4.64)} && (4.70)\\ &= \int_{\mathbb{R}} \hat{\phi}(\xi)d\xi \;\; \text{by def.of } L_1 && (4.71)\\ &= 2\pi\phi(0) \;\; \text{by the Fourier inversion formula} && (4.72) \end{aligned}$$

Notice that this time the last step uses something deep. ■

Consistency of the two definitions of the Fourier transform. If $f \in L^1$ then we already have a meaning for $\hat{f}$. So is

$$\hat{L}_f = L_{\hat{f}}?$$

Yes. Here is a (definition chasing) proof.

$$\begin{aligned}
\hat{L}_f(\phi) &= L_f(\hat{\phi}) && (4.73)\\
&= \int_{\mathbb{R}} f(\xi)\hat{\phi}(\xi)d\xi && (4.74)\\
&= \int_{\mathbb{R}} f(\xi)\int_{\mathbb{R}} e^{ix\xi}\phi(x)dxd\xi && (4.75)\\
&= \int_{\mathbb{R}} \big(\int_{\mathbb{R}} f(\xi)e^{ix\xi}d\xi\big)\phi(x)dx && (4.76)\\
&= \int_{\mathbb{R}} \hat{f}(x)\phi(x)dx && (4.77)\\
&= L_{\hat{f}}(\phi) && (4.78)
\end{aligned}$$

QED

4.4 Problems on the Fourier Transform

1. Find the Fourier transforms of the following functions:

   a) $e^{-|x|}$

   b) $e^{-x^2/2}$

   c) $xe^{-x^2/2}$

   d) $x^2e^{-x^2/2}$

   e) $f(x) = \begin{cases} 1 & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$

   f) $\dfrac{1}{1+x^2}$

2. For φ in $\mathcal{S}(R^2)$ define

$$T(\varphi) = \int_{-\infty}^{\infty} x\varphi(x,0)dx.$$

(Note that this is an integral over a line — not over R^2.)

a) Show that T is in $\mathcal{S}'(R^2)$.

b) Find the Fourier transform, $\widehat{T}$, of T explicitly (explicitly enough to do part c) without finding $\widehat{\psi}$).

c) Evaluate $\widehat{T}(\psi)$ where

$$\psi(\xi,\eta) = \frac{\xi e^{-(\xi^2+\eta^2)}}{1+\xi^2}$$

d) Let $L = a\dfrac{\partial^2}{\partial\xi^2} + b\dfrac{\partial^2}{\partial\xi\partial\eta} + c\dfrac{\partial^2}{\partial\eta^2}$. Find all values of the real parameters a, b, c such that

$$L\widehat{T} = 0.$$

3. Compute

   a. $\widehat{\delta'}$

   b. $\widehat{\delta''}$

   c. $\hat{L}_x$

d. $\hat{L}_{x^2}$

Hints: 1. Use the definitions. 2. *Use the definitions.* 3. **Use the definitions**.

4. Show that a function f on R^n is rapidly decreasing if and only if, for every polynomial $p(x_1, \ldots, x_n)$, the function $p(x)f(x)$ is bounded.

5. Derive the identities (4.59) - (4.61) from the one-dimensional identities (4.14) - (4.16).

5 Applications of the Fourier Transform

5.1 The Heat Equation by Fourier transform.

The three fundamaental types of PDE, parabolic, elliptic and hyperbolic are amenable to study by using the Fourier transform in slightly differing ways for each. The prototype of these three kinds of PDEs are the heat equaiton, Poisson equation and wave equation.

The heat equation for a function $u = u(t, x)$ in n spatial dimensions is

$$\partial u/\partial t = (1/2)\Delta u \quad t \geq 0 \tag{5.1}$$

9.1

with initial condition

$$u(0, x) = f(x). \tag{5.2}$$

9.2

Here f is a given function on $\mathbb{R}^n$. As you probably know very well, there is strong physical motivation for seeking a solution to (5.1) for $t \geq 0$ only: bodies cool off as time moves forward. If you move backward in time temperatures can increase in such a way as to produce singularities in the solution. We will see on purely mathematical grounds why solving the equation (5.1) backwards in time can produce singularities.

Solution of (5.1), (5.2): Apply the Fourier transform to u in the spatial variable only. We obtain a function $\hat{u}(t, \xi)$ satisfying

$$\partial \hat{u}(t, \xi)/\partial t = -(1/2)|\xi|^2 \hat{u}(\xi) \tag{5.3}$$

9.3

which we would like to satisfy the intial condition

$$\hat{u}(0, \xi) = \hat{f}(\xi) \tag{5.4}$$

9.4

Now for each $\xi \in \mathbb{R}^n$ the equation (5.3) an ORDINARY differential equation, whose solution we know from elementary calculus. It is

$$\hat{u}(t, \xi) = e^{-(t/2)|\xi|^2} \hat{f}(\xi). \tag{5.5}$$

9.5

In order to find $u(t, x)$ itself we still have to do a reverse Fourier transform on (5.4). But Hey! We already know of a function whose Fourier transform is the first factor on the right in (5.4). It is the function

$$p_t(x) = (2\pi t)^{-n/2} e^{-|x|^2/(2t)} \tag{5.6}$$

9.6

whose Fourier transform we already computed. See (F10 4.13) and thereafter. (Use the n-dimensional version of that function.)

We also know that convolution goes over to multiplication under Fourier transformation. Hence

$$(p_t * f)\hat{}(\xi) = \hat{p}_t \hat{f}(\xi) \tag{5.7}$$
$$= e^{-(t/2)|\xi|^2} \hat{f}(\xi). \tag{5.8}$$

By the uniqueness property of the Fourier transform we therefore find

$$u(t,x) = (p_t * f)(x) \tag{5.9}$$
$$= \frac{1}{\sqrt{(2\pi t)^n}} \int_{\mathbb{R}^n} e^{-|x-y|^2/2t} f(y) d^n y \tag{5.10}$$

5.2 Poissson's equation by Fourier transform

We wish now to apply the preceding method to the equation

$$\Delta u = -4\pi\rho. \tag{5.11}$$ 9.10

Although the method works in n dimensions (with $n \geq 2$), we will carry this out only in dimension $n = 3$, where we can compare with our previous solution.

Apply the Fourier transform to (9.10 5.11) in ALL of the variables. (Of course we only have spatial variables this time.) We find

$$-|\xi|^2 \hat{u}(\xi) = -4\pi\hat{\rho}(\xi) \tag{5.12}$$ 9.11

This time the partial differential equation has been reduced to an algebraic equation: there are no derivatives left at all. So you think that the solution is

$$\hat{u}(\xi) = -4\pi(1/|\xi|^2)\hat{\rho}(\xi)? \tag{5.13}$$ 9.12

Well, if that were right then we again have a solution for the Fourier transform in the form of a product. Of course this will give u itself as a convolution. In fact it is true that

$$\widehat{(\frac{1}{r})}(\xi) = -4\pi/|\xi|^2$$

And this would give

$$u = \frac{1}{r} * \rho, \tag{5.14}$$

which we already know from our previous work is actually correct. We saw back in those days that ρ could be e.g any charge distribution in $\mathcal{D}^*$ with compact support. Such a distribution ρ is clearly in $\mathcal{S}^*$ also and the computations (5.12)-(5.14) are actually correct.

But we are losing some solutions in the passage from (5.12) to (5.13)! For example we have the identity

$$|\xi|^2 \delta(\xi) = 0. \tag{5.15}$$

This means that we could add $5\delta(\xi)$ onto the right side of (5.13) and get another solution. Back in x space this just means that we can add a constant onto any solution u and get another solution. In fact if v as any harmonic function on $\mathbb{R}^3$ (i.e. $\Delta v = 0$) we could add v onto any solution of (5.11) and get another solution. For example $3x + 2y + 9z$ is harmonic. So is $x^2 + y^2 - 2z^2$. There are, in fact, lots of harmonic polynomials on $\mathbb{R}^3$. Among all the solutions to (5.11) the solutions (5.14) are the only ones that go to zero as $|x| \to \infty$. The solution (5.14) is called the *potential* of ρ. It is true that all the solutions of (5.12) differ from (5.13) only in such a way. Here is a problem that illuminates this.

Problem:(Poisson by Fourier)

Suppose that $p(x, y, z)$ is a harmonic polynomial on $\mathbb{R}^3$. That is, $\Delta p = 0$. Write $\partial_j = \partial/\partial\xi_j$ and let L be the distribution

$$L = p(-i\partial_1, -i\partial_2, -i\partial_3)\delta.$$

Show that

$$|\xi|^2 L = 0.$$

Recall that this means $L(|\xi|^2\phi(\xi)) = 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^{\ni})$.

5.3 Advanced, Retarded and Feynman Propagators for the forced harmonic oscillator

Let ω be a strictly positive constant. The harmonic oscillator equation with frequency ω (actually frequency $= \omega/2\pi$) is

$$\frac{d^2u(t)}{dt^2} + \omega^2 u(t) = 0. \tag{5.16}$$ 10.1

the general solution to (10.1) is

$$u(t) = Ae^{i\omega t} + Be^{-i\omega t}. \tag{5.17}$$ 10.2

where A and B are complex constants. Of course one can choose A and B so that the solution is real and is expressible as a linear combination of $\sin \omega t$ and $\cos \omega t$. But we are going to stick to the complex form (5.17) because it will be more revealing for our purposes.

The *forced* harmonic oscillator equation is

$$\frac{d^2u(t)}{dt^2} + \omega^2 u(t) = f(t). \tag{5.18}$$ 10.3

It will be adequate for our purposes just to assume that $f \in C_c^\infty(\mathbb{R})$ and then not have to worry about any technical details. We can think of a weight hanging from a spring which is attached to the ceiling. $u(t)$ is the displacement of the weight from its neutral position. For a few seconds we push and pull on the weight (up and down) in accordance with the force $f(t)$ and then let go altogether (at the upper bound of the support set of f.) What happens to the weight during and after the disturbance (thats us) acts via f? Of course that depends on the initial state of the weight (at time $t = 0$, say). Was the weight in its neutral position ($u = 0$) or was it elsewhere? Was it moving or stationary? If we specify the intial position $u(0)$ and the initial velocity $u'(0)$ then we must solve the Initial Value Problem for the equation (5.18) with the specified f and specified initial data. Our goal however is to address a different problem. We are going to study the equation (5.18) from the point of view of the Boundary Value Problem, somewhat in the spirit of the section on Green functions. The novelty of the present section lies in the fact that we will be interested in *boundary conditions at* $\pm\infty$. The function f is sometimes called the *source* generating the motion u. For example in the context of Maxwell's equations the sources for the electric and magnetic fields are charges and currents. As in (5.18) they make the homogeneous Maxwell equations inhomogeneous.

We saw in the section on Green functions that on an interval $[a, b]$ one can choose, at a, Dirichlet or Neumann conditions or a linear combination of them. Similarly at b. In this sense we have four degrees of freedom in the nature of our choice of boundary conditions. (But you can only impose two of them on a solution.) Here are three different Green functions for the equation (5.18). We will discuss their physical interpretations afterward. Let

$$G_r(t) = \begin{cases} \frac{\sin t\omega}{\omega} & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{5.19}$$

10.5

$$G_a(t) = \begin{cases} 0 & t \geq 0 \\ -\frac{\sin t\omega}{\omega} & t < 0 \end{cases} \tag{5.20}$$

10.6

$$G_F(t) = \frac{1}{2i\omega} \begin{cases} e^{it\omega} & t \geq 0 \\ e^{-it\omega} & t < 0 \end{cases} \tag{5.21}$$

Problem 1. Suppose that $f \in C_c^\infty(\mathbb{R})$. Let

$$u = G * f$$

with $G = G_a$ or G_r or G_F.

a. Show that in each case u is a solution to (5.18). Hint. Imitate the proof of Theorem 2.7. Note that each of these three functions is continuous and has a jump of one in its first derivative at $t = 0$.

b. Show that the solution $G_r * f$ is zero in the "distant" past, i.e. for sufficiently large negative time, depending on f.

c. Show that $G_a * f$ is zero in the "distant" future.

Terminology. The Green function G_r is called the *retarded* Green function, and also sometimes called the *retarded propagator*. It propagates the disturbance f into the future and depends on the the disturbance at some earlier time. The etymology of the word "retarded" can be understood in the context of electromagnetic theory, where the potential,at $\mathbf{x}$ t, of a changing charge distribution depends not on the charge distribution at time t but at an earlier time. The field produced by the charge distribution travels only with the speed of light, not instantly. The resulting potential is called the retarded potential.

Similary the propagator G_a is called the *advanced propagator*. As you saw in part c. of Problem 1, $u_a(t)$ can be non-zero before the disturbance f even begins!!! (Doesn't sound very "causal", does it?) One says that G_a propagates the disturbance into the past.

The third propagator, G_F, is particularly important in understanding the behavior of electrons and positrons in combination. G_F is called the *Feynman propagator* (for the single frequency ω.) It propagates the disturbance f into the future as a positive frequency wave and into the past as a negative frequency wave. When this discussion is boosted up to three space dimensions (from the present zero space dimensions) Feynman's propagator has the interpretation of propagating electron wave functions forward in time as positive energy wave functions and propagating positron wave functions backward in time as negative energy wave functions. Some authors say, for short, that positrons are negative energy electrons moving backward in time.

The Green functions above have been constructed directly in terms of the homogeneous solutions in accordance with the method in the section on Green functions. Here is how the Fourier transform method can be applied. In some contexts its the more useful way to go.

Take the Fourier transform of (5.16) in the t variable (which is the only variable around, this time.) We will use s for the variable conjugate to t. That is, we define $\hat{u}(s) = \int_{-\infty}^{\infty} e^{ist}u(t)dt$. We find

$$(-s^2 + \omega^2)\hat{u}(s) = \hat{f}(s). \tag{5.22}$$ [10.11]

For the homogeneous equation we need to put $f = 0$. The solutions to (5.22) then include the δ functions at the two roots of $\omega^2 - s^2$. Thus

$$\hat{u}(s) = A\delta(s - \omega) + B\delta(s + \omega) \tag{5.23}$$ [10.12]

gives solutions to the homogeneous equation (on the Fourier transform side.) It is a FACT that these are the only solutions to (5.22) when $f = 0$.

Problem 2. Show that the Fourier transform of the solutions (5.17) are given by (5.23) (up to a constant.) Thus we have recovered the solutions to the homogeneous equation by the method of Fourier transforms.

Next we address the inhomogeneous equation. Note first the identity

$$\frac{1}{s^2 - \omega^2} = \frac{1}{2\omega}\Big(\frac{1}{s - \omega} - \frac{1}{s + \omega}\Big) \tag{5.24}$$ [10.13]

Dividing (10.11 5.22) by the coefficient of $\hat{u}(s)$ we find

$$\hat{u}(s) = -\frac{1}{2\omega}\left(\frac{\hat{f}(s)}{s-\omega} - \frac{\hat{f}(s)}{s+\omega}\right) \tag{5.25}$$ 10.14

TROUBLE: We already know that $\frac{1}{s\pm\omega}$ has a non-integrable singularity and therefore so does the right hand side of (10.14 5.25). As it stands the right hand side therefore has no meaning. But we saw earlier that the function $1/s$ has an interpretation as a distribution called the Principle Part of $1/s$. It is defined as

$$P(\frac{1}{s}) < \phi >= \lim_{c\to 0}\int_{|s|>c} \frac{\phi(s)}{s} \tag{5.26}$$ 10.15

Lemma 2. Let

$$\theta(t) = (1/2)sgn(t) \tag{5.27}$$ 10.16

Then

$$\hat{\theta}(s) = iP\frac{1}{s} \tag{5.28}$$ 10.17

Proof:

$$\begin{aligned}
\hat{\theta}(\phi) &= \theta(\hat{\phi}) && (5.29)\\
&= \lim_{a\to\infty}\int_{-a}^{a} \theta(t)\hat{\phi}(t)dt && (5.30)\\
&= \lim_{a\to\infty}\int_{-a}^{a}\left(\int_{-\infty}^{\infty} \theta(t)e^{itx}\phi(x)dx\right)dt && (5.31)\\
&= \lim_{a\to\infty}\int_{-\infty}^{\infty}\left(\int_{-a}^{a} \theta(t)e^{itx}dt\right)\phi(x)dx && (5.32)\\
&= \lim_{a\to\infty}\int_{-\infty}^{\infty}\left(\frac{\cos ax - 1}{ix}\right)\phi(x)dx && (5.33)\\
&= \lim_{a\to\infty}\int_{|x|>1}\left(\frac{\cos ax - 1}{ix}\right)\phi(x)dx && (5.34)\\
&+ \lim_{a\to\infty}\int_{|x|\le 1}\left(\frac{\cos ax - 1}{ix}\right)(\phi(x)-\phi(0))dx && (5.35)
\end{aligned}$$

because $(\frac{\cos ax - 1}{ix})$ is an odd function. The next to the last line converges to $\int_{|x|>1}(-1)\frac{\phi(x)}{ix}dx$ by the Riemann-Lebesgue Lemma. The very last integral may be rewritten $\int_{|x|\leq 1}(\cos ax - 1)\frac{\phi(x)-\phi(0)}{ix}dx$. So this integral converges to $-\int_{-1}^{1}\frac{\phi(x)-\phi(0)}{ix}dx$ by the Riemann-Lebesgue Lemma also. Q.E.D.

Problem 3. Using Lemma 2 show that

$$(\theta(t)e^{it\omega})\hat{}(s) = iP\frac{1}{s+\omega} \tag{5.36}$$ 10.17

for any real ω.

Problem 4 Let

$$F^{\pm}(t) = \theta(t)e^{\pm it\omega} \quad \text{for } \omega > 0 \tag{5.37}$$ 10.18

In view of ([10.14] 5.25) and ([10.17] 5.36) we should expect that all the solutions to

$$\frac{d^2u(t)}{dt^2} + \omega^2 u(t) = \delta(t). \tag{5.38}$$ 10.19

should be expressible as a linear combination of the functions $F^{\pm}$ and the solutions to the homogeneous equation ([10.1] 5.16). For example one expects

$$G_r(t) = AF^{+}(t) + BF^{-}(t) + Ce^{i\omega t} + De^{-i\omega t} \tag{5.39}$$

for some constants A, B, C, D.

Find explicitly this representation of the three Green functions G_r, G_a and G_F in terms of the functions $F^{\pm}$ and the solutions, ([10.2] 5.17), of the homogeneous equation.

5.4 The wave equation by Fourier transform

The physically appropriate problem is the Cauchy problem. This is the initial value problem.

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \Delta u(t,x) \tag{5.40}$$
9.30

$$u(0,x) = f(x) \tag{5.41}$$

$$\partial u(0,x)/\partial t = g(x) \tag{5.42}$$
9.31

As in the heat equation we will take the Fourier transform in the space variables only. We find

$$\frac{\partial^2 \hat{u}(t,\xi)}{\partial t^2} = -|\xi|^2 \hat{u}(t,\xi) \tag{5.43}$$
9.32

Just as in the case of the heat equation we have now an ordinary differential equation. But this time its second order in t. The general solution to (5.43) for each ξ is

$$\hat{u}(t,\xi) = A(\xi)\cos t|\xi| + B(\xi)\sin t|\xi| \tag{5.44}$$
9.33

Upon Fourier transforming the initial conditions (5.42) we see that we must have $A(\xi) = \hat{f}(\xi)$ and $|\xi|B(\xi) = \hat{g}(\xi)$. Hence the Fourier transform of u is given by

$$\hat{u}(t,\xi) = \hat{f}(\xi)\cos t|\xi| + \hat{g}(\xi)\frac{\sin t|\xi|}{|\xi|}. \tag{5.45}$$
9.34

That was the easy part. Now we have to Fourier transform back to find u itself. Notice first that $\cos t|\xi|$ is exactly the t derivative of $(\sin t|\xi|)/|\xi|$. It will suffice then to find a distribution ν_t on $\mathbb{R}^3$ whose Fourier transform, for each t, is $(\sin t|\xi|)/|\xi|$. The solution to (5.40), (5.42) is then clearly

$$u(t,x) = (\nu_t * g)(x) + (\partial/\partial t)(\nu_t * f)(x). \tag{5.46}$$
9.35

Lemma 9.5 Let σ_t be surface area element on the sphere $|x| = |t|$ in $\mathbb{R}^3$. Then

$$\int_{\mathbb{R}^3} e^{ix\cdot\xi} d\sigma_t(x) = 4\pi t \frac{\sin t|\xi|}{|\xi|} \tag{5.47}$$
9.36

Proof. The integral is just an integral over a sphere of radius $r = |t|$. Use spherical polar coordinates ϕ, θ. We can make use of the rotation invariance of the integral by choosing the positive z axis in the direction of ξ. Let $a = |\xi|$. Then $x \cdot \xi = ra\cos\theta$. So the integral is

$$r^2 \int_0^{2\pi} \int_0^{\pi} e^{ira\cos\theta} \sin\theta d\theta d\phi = 4\pi r \frac{\sin ra}{a}$$

(which you get by substituting $s = \cos\theta$.) This proves (5.47) once one observes that even if $t < 0$ (5.47) is correct because sin is odd. QED

Now define a distribution ν_t on $\mathbb{R}^3$ by the formula

$$\nu_t(\phi) = \frac{1}{4\pi|t|} \int_{|x|=|t|} \phi(x) d\sigma_t(x) \text{ for } \phi \in \mathcal{S}(\mathbb{R}^3) \tag{5.48}$$ 9.38

for $t \neq 0$ and define ν_t to be zero for $t = 0$.

5.4.1 Problems

Problem 1. Prove that

$$\hat{\nu}_t(\xi) = \frac{\sin t|\xi|}{|\xi|} \text{ for all } t \tag{5.49}$$ 9.39

Hints: 1. Use the definition of Fourier transform of a distribution.

2. Use the definition of ν_t.

3. Use Lemma 9.5

Problem 2 Explain why the solution (5.46), in combination with the explicit form (5.48) of ν_t, says that light travels with exactly speed one (in our units).

Finally, lets consider the inhomogeneous wave equation

$$\partial^2 u(t,x)/\partial t^2 - \Delta u(t,x) = \rho(t,x). \tag{5.50}$$ 9.40

Assume, for ease of mind, that ρ is in $C_c^\infty(\mathbb{R}^4)$. If we first Fourier transform in the spatial variables only we find

$$\partial^2 \hat{u}(t,\xi)/\partial t^2 + |\xi|^2 \hat{u}(t,\xi) = \hat{\rho}(t,\xi) \tag{5.51}$$ 9.41

Notice that for each ξ this equation looks just like the harmonic oscillator equation (5.16) with $\omega^2 = |\xi|^2$. !!! We can, and will, take over what we learned from the forced harmonic oscillator. It will be sufficiently illuminating to focus on just one of the three propagators that we studied for the harmonic oscillator.

Put $\omega = |\xi|$ in (5.36) and do the usual reverse Fourier transform to find.

$$\hat{u}(t,\xi) = (\theta(t)\frac{e^{it|\xi|}}{2|\xi|} - \theta(t)\frac{e^{-it|\xi|}}{2|\xi|}) * \hat{\rho}(t,\xi)$$

For each t and t' we see that we have the usual product of Fourier transforms on the right (in the spatial variables). Moreover we have the formula (5.49) at our disposal. Thus we find

$$u(t,x) = \int_{-\infty}^{\infty} (\theta(t-t')(\nu_{t-t'} * \rho(t',\cdot))(x)dt' \tag{5.52}$$

9.50

Not only does the integral represent a convolution but there is also a convolution right in the middle of the integral. All in all this represents a convolution over $\mathbb{R}^4$. We may write this as

$$u = G * \rho \tag{5.53}$$

9.51

where

$$G = \theta(t)\nu_t \tag{5.54}$$

9.52

is a distribution on $\mathbb{R}^3$ for each t and all together is a distribution on $\mathbb{R}^4$. This is one of the propagators for the inhomogeneous wave equation (9.40).

One might wish to write (5.54) more suggestively to emphasize its character as a distribution over $\mathbb{R}^4$ as $G(t,x) = \theta(t)\nu_t(x)$. But one should not lose sight of the fact that the second factor is not actually a function.

Remark to ponder. For the wave equation, as for the forced harmonic oscilator, there are "essentially" three propagators for the inhomogeneous equation (5.50). We arrived at one of them by the above procedure. Its one half the advanced plus retarded. Where does the Feynman propagator come from? Hint: Consider the identity $\lim_{\epsilon\downarrow 0} \frac{1}{x+i\epsilon} = P(1/x) + \pi i\delta$ which you proved long ago in (3.17).

6 Laplace Transform

Definition. The Laplace transform of a function $f : [0,\infty) \to \mathbb{C}$ is the function L_f defined by

$$L_f(s) = \int_0^\infty f(t)e^{-st}dt. \tag{6.1}$$ L1

The domain of L_f is the set $\mathcal{D}(L_f)$ consisting of those $s \in \mathbb{R}$ for which $\int_0^\infty |f(t)e^{-st}|dt < \infty$.

Examples. 1. $f(t) = e^{t^2}$, $\mathcal{D}(L_f) = \phi$

2. If $f \in L^1(0,\infty)$ then $\mathcal{D}(L_f) \supset [0,\infty)$. E.g., if $f(t) = 1/(1+t^2)$ then $\mathcal{D}(L_f) = [0,\infty)$.

3. $f(t) = e^{3t}$. Then $L_f(s) = \int_0^\infty e^{(3-s)t}dt < \infty$ if $s > 3$. So $\mathcal{D}(L_f) = (3,\infty)$.

4. $f(t) = e^{-t^2}$. Then $\mathcal{D}(L_f) = (-\infty,\infty)$. Note. If $s_0 \in \mathcal{D}(L_f)$ then $s \in \mathcal{D}(L_f)$ for any $s > s_0$ because

$$f(t)e^{-st} = \underbrace{f(t)e^{-s_0t}}_{\in L^1(0,\infty)} \underbrace{e^{-(s-s_0)t}}_{\text{bdd}}$$

Consequently $\mathcal{D}(L_f)$ is always an interval. As we see from the above examples it may be the empty set or of the form $[s_0,\infty)$, or (s_0,∞), or $(-\infty,\infty)$.

Remark. If $s_0 \in \mathcal{D}(L_f)$ then L_f has an analytic extension to the half space Re $z > s_0$.

Proof: Put $z = s + i\tau$ with $s > s_0$. Then

$$|e^{-zt}f(t)| = e^{-st}|f(t)| \in L^1(0,\infty).$$

Hence the function

$$\widetilde{L}(z) = \int_0^\infty f(t)e^{-zt}dt \tag{6.2}$$ L2

exists for Re $z > s_0$.

By differentiating under the integral sign (easily justified) we see that $\widetilde{L}(z)$ is analytic with

$$d\widetilde{L}(z)/dz = \int_0^\infty -tf(t)e^{-zt}dt.$$

Example. $f(t) = e^{3t}$, $\mathcal{D}(L_f) = (3, \infty)$

$$\widetilde{L}(z) = \int_0^\infty e^{(3-z)t}dt = \frac{1}{z-3}$$

which is analytic in $\mathrm{Re}(z) > 3$. But note that $\widetilde{L}$ has an analytic extension to the entire complex plane with the exception of a pole at $z = 3$. This is typical of Laplace transforms that arise in practice. They often have meromorphic extensions to the entire plane. But the integral representation may be only valid for Re $z > s_0$.

6.1 The Inversion Formula

Suppose $s_0 \in \mathcal{D}(L_f)$ and $s_1 > s_0$. Let

$$\begin{aligned} g(\tau) &= \widetilde{L}(s_1 + i\tau) && (6.3) \\ &= \int_0^\infty f(t)e^{-(s_1+i\tau)t}dt && (6.4) \\ &= \int_0^\infty \underbrace{\left(f(t)e^{-s_1 t}\right)}_{\in L^1(0,\infty)} e^{-i\tau t}dt. && (6.5) \end{aligned}$$

Put

$$h(t) = \begin{cases} 0 & -\infty < t < 0 \\ f(t)e^{-s_1 t} & 0 \le t < \infty. \end{cases}$$

Then

$$g(\tau) = \int_{-\infty}^\infty h(t)e^{-i\tau t}dt = \text{Fourier transform of } h.$$

Hence

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^\infty g(\tau)e^{i\tau t}d\tau.$$

This integral may converge pointwise (in t) or in the L^2 sense of convergence, depending on the quality of g. But we ignore these questions.

Thus for $0 \le t < \infty$

$$f(t)e^{-s_1 t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\tau) e^{i\tau t} d\tau$$

by the Fourier inversion formula. So

$$f(t) = \frac{e^{s_1 t}}{2\pi} \int_{-\infty}^{\infty} \widetilde{L}(s_1 + i\tau) e^{i\tau t} d\tau \quad \text{for} \quad t \ge 0. \tag{6.6}$$ L3

Corollary 6.1 *(Uniqueness Theorem) The Laplace transform $L_f(s)$ of a function f on $[0, \infty)$ determines f uniquely (up to values on a set of measure zero) if $\mathcal{D}(L_f)$ is not empty.*

6.2 Application of Uniqueness

Problem. Find the function $f(t)$ whose Laplace Transform is

$$L(z) = \frac{4}{z-5} + \frac{1}{z-7} + 8\frac{z}{z^2+9}.$$

Solution: $f(t) = 4e^{5t} + e^{7t} + 8\cos 3t$.

Reason.

$$z/(z^2+9) = +\frac{1}{2}\left[\frac{1}{z-3i} + \frac{1}{z+3i}\right].$$

Note. In this problem $L(z)$ had simple poles. Higher order poles can be dealt with using

$$\frac{d\widetilde{L}(z)}{dz} = \int_0^{\infty} -te^{-zt} f(t) dt.$$

But note. $\widetilde{L}$ may have no singularities in the finite plane even though f is not the zero function.

Example. Let f be in $L^1(R)$ with support in $[0, a]$. Then

$$\widetilde{L}(z) = \int_0^{\infty} f(t)e^{-zt} dt = \int_0^{a} f(t) e^{-zt} dt$$

exists for all $z \in \mathbb{C}$ and is entire.

Nevertheless the case where $\widetilde{L}$ has only poles and goes to zero at ∞ is important. For this situation it is useful to rewrite the inversion formula thus:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{L}(s_1 + i\tau) e^{(s_1+i\tau)t} d\tau.$$

Putting $z = s_1 + i\tau$, $dz = i d\tau$ we get

$$f(t) = \frac{1}{2\pi i} \int_{s_1 - i\infty}^{s_1 + i\infty} \widetilde{L}(z) e^{zt} dz.$$

Sometimes the straight line integration contour can be deformed to surround the poles of $\widetilde{L}$. See the homework problem.

6.3 Problems on the Laplace Transform

1. We have seen that if

$$\widetilde{L}(z) \approx \int_0^\infty f(t)e^{-zt}dt$$

exists in the open half space Re $z > s_0$ then for any real number $s_1 > s_0$ f may be expressed by

$$f(z) = \frac{e^{s_1 t}}{2\pi} \int_{-\infty}^{\infty} \widetilde{L}(s_1 + i\tau)e^{i\tau}d\tau \qquad (6.7) \quad \boxed{\texttt{L10}}$$

with $z = s + i\tau$. This may clearly be rewritten

$$f(t) = \frac{1}{2\pi i} \int_C \widetilde{L}(z)e^{zt}dz \qquad (6.8) \quad \boxed{\texttt{L11}}$$

where C is the straight line contour from $s_1 - i\infty$ to $s_1 + i\infty$.

For some functions $\widetilde{L}(z)$ it may be possible to deform the contour to C_1: so as to include all the singular points of $\widetilde{L}$ inside. If these singular points are isolated poles then one could find $f(z)$ by the method of residues.

Justify this procedure for the function

$$\widetilde{L}(z) = \frac{4}{(5-z)^3} + \frac{2}{6-z} + \pi\frac{z^2 - 2z + 9}{(z^2+9)^2}$$

and use it to find $f(t)$ for all $t \geq 0$.

7 Asymptotic Expansions

Reference: C.M. Bender and A.S. Orszag, Advanced Mathematical Methods for Scientists and Engineers, Chapter 6. [QA 371 B45]

7.1 Introduction

Definition If f and g are two real (or complex) valued functions on a set S then one writes

$$f = O(g) \quad \text{on } S$$

if there is a real constant C such that

$$|f(x)| \leq C|g(x)| \quad \text{for all} \quad x \in S.$$

Example. $x(1+x^2)^{1/2} = O(x^2)$ on $[1, \infty)$ but not on $[0, \infty)$
Definition If f and g are two functions on an interval (a, ∞) then one writes

$$f = O(g) \quad \text{as} \quad x \to \infty$$

if $f = O(g)$ on $[M, \infty)$ for some M.
Example $x(1+x^2)^{1/2} = O(x^2)$ as $x \to \infty$.
Definition $f(x) = o(g(x))$ as $x \to \infty$ means

$$\lim_{x\to\infty} f(x)/g(x) = 0.$$

Examples. (i) If $f(x) = 1/(x^2+1)$and $g(x) = 1/x$ then $f(x) = o(g(x))$ as $x \to \infty$.
(ii) $e^{2x} = o(e^{3x})$ as $x \to \infty$.
More Notation $f(x) = \phi(x) + o(\psi(x))$ as $x \to \infty$ means that $f(x) - \phi(x) = o(\psi(x))$ as $x \to \infty$.
Example.

$$\frac{1}{x^2+1} = \frac{1}{x^2} + o(1/x^2) \quad \text{as} \quad x \to \infty.$$

We are going to study the four basic methods for determining the

Asymptotic behavior of an integral.

Four Methods
1. Integration by parts [page 45 of NB]
2. Laplace's method [p.47–53 of NB]
3. Stationary phase [p.89–91 of NB]
4. Steepest descent [p.92–100 of NB]

7.2 The Method of Integration by Parts.

PROBLEM: Let

$$f(x) = \int_1^x e^t/t dt, \quad x > 1.$$

Find the asymptotic behavior of $f(x)$ as $x \to \infty$.
SOLUTION: Integrate by parts to find

$$f(x) = \int_1^x (1/t) de^t \tag{7.1}$$

$$= \frac{e^t}{t}|_1^x + \int_1^x \frac{e^t}{t^2} dt \tag{7.2}$$

$$= \frac{e^x}{x} - e + \int_1^x \frac{e^t}{t^2} dt \tag{7.3}$$

Now it happens that the last term (the integral) is rather small compared to the first term for large x. This needs to be proved. But once it is proved we can see that $f(x)$ "behaves" like e^x/x for large x, the precise meaning of which can be stated in terms of the preceeding notation in the form

$$f(x) = \frac{e^x}{x} + o(\frac{e^x}{x}) \quad \text{as} \quad x \to \infty.$$

We will do better than this. But first we need the following tricky lemma.

Lemma 7.1 *For any integer $n \geq 1$ we have*

$$\int_1^x \frac{e^t}{t^n} dt = O(\frac{e^x}{x^n}) \quad \textit{as} \quad x \to \infty.$$

Proof:

$$\int_1^x \frac{e^t}{t^n}dt = \int_1^{x/2} \frac{e^t}{t^n}dt + \int_{x/2}^x \frac{e^t}{t^n}dt \tag{7.4}$$

$$\leq \int_1^{x/2} e^t dt + \int_{x/2}^x \frac{e^t}{(x/2)^n}dt \tag{7.5}$$

$$= e^{x/2} - e + (2/x)^n(e^x - e^{x/2}), \tag{7.6}$$

which proves the lemma.

Corollary 7.2

$$\int_1^x \frac{e^t}{t^{(n+1)}}dt = o(\frac{e^x}{x^n}) \quad as \quad x \to \infty.$$

One can go a step further now in determining the asymptotic behavior of f. Integrating by parts again

$$f(x) = \frac{e^t}{t}\Big|^x + \frac{e^t}{t^2}\Big|_1^x + 2\int_1^x \frac{e^t}{t^3}dt = e^x\Big(\frac{1}{x} + \frac{1}{x^2}\Big) + O\Big(\frac{e^x}{x^3}\Big) \text{ as before.}$$

Clearly we may continue this and get

$$f(x) = e^x\big(\frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{6}{x^4} + \cdots + \frac{(n-1)!}{x^n} + o\Big(\frac{1}{x^n}\Big)\Big).$$

This motivates the following definition.

Definition. Let $\varphi_1, \varphi_2, \varphi_3, \ldots$ be a sequence of functions on $[a, \infty)$ such that $\varphi_{n+1}(x) = o(\varphi_n(x))$. If f is a function on $[a, \infty)$ one says that the series $\sum_{j=1}^{\infty} a_j\varphi_j(x)$ (a_j are constants) is an *asymptotic expansion* of f if for each $n = 1, 2, 3, \ldots$

$$f(x) = \sum_{j=1}^{n} a_j\varphi_j(x) + o(\varphi_n(x)) \quad x \to \infty.$$

Thus the infinite series $\sum_{j=1}^{\infty} a_j\varphi_j(x)$ *need not converge* for any x. In the preceding example the series $\sum_{n=1}^{\infty} \frac{(n-1)!}{x^n}$ is easily seen (by the ratio test) to converge for no x.

7.3 Laplaces Method.

7.3.1 Watson's Lemma

Theorem 7.3 *(Watson's Lemma) Assume*

A. $\int_0^\infty |f(t)|e^{-ct}dt < \infty$ *for some* $c > 0$.

B. For some $\alpha > -1$ *and* $\beta > 0$, f *has the asymptotic expansion*

$$f(t) = t^\alpha \sum_{k=0}^{\infty} a_k t^{k\beta} \text{ as } t \downarrow 0. \tag{7.7}$$ As1

Let

$$I(x) = \int_0^\infty f(t)e^{-xt}dt \qquad x > c. \tag{7.8}$$ As2

Then $I(x)$ *has the asymptotic expansion*

$$I(x) = \sum_{k=0}^{\infty} \frac{a_k \Gamma(\alpha + k\beta + 1)}{x^{\alpha+k\beta+1}} \qquad x \to \infty. \tag{7.9}$$ As3

where

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du \text{ for } x > 0.$$

Proof: First, note that for a simple power we have

$$\int_0^\infty t^{\gamma-1}e^{-xt}dt = x^{-\gamma}\int_0^\infty s^{\gamma-1}e^{-s}ds \text{ by putting } s = xt \tag{7.10}$$

$$= x^{-\gamma}\Gamma(\gamma), \;\; \gamma > 0. \tag{7.11}$$ As4

Second, note that we may replace $\int_0^\infty f(t)e^{-x}$ by $\int_0^\delta f(t)e^{-x}$ and incur an exponentially small error as in the following lemma.

Lemma 7.4 *For any* $\delta > 0$ *and* f *satisfying A*

$$\int_\delta^\infty f(t)e^{-xt}dt = o(x^{-\gamma}) \text{ as } x \to \infty \text{ for all } \gamma > 0. \tag{7.12}$$ As5

Proof of Lemma 1: For $x > c$ we have

$$\left|\int_\delta^\infty f(t)e^{-xt}dt\right| \leq \int_\delta^\infty |f(t)|e^{-ct}e^{-(x-c)t}dt \tag{7.13}$$

$$\leq e^{-(x-c)\delta}\int_\delta^\infty |f(t)|e^{-ct}dt = o(x^{-\gamma}) \quad x \to \infty \ \forall\ \gamma > 0. \tag{7.14}$$

As7

Q.E.D.

Now in order to prove the theorem we choose N and let $\gamma = \alpha + N\beta + 1$. We must show that

$$I(x) - \sum_{k=0}^{N} \frac{a_k \gamma(\alpha + k\beta + 1)}{x^{\alpha+k\beta+1}} = o(x^{-\gamma}) \quad x \to \infty \tag{7.15}$$

As8

since $x^{-\gamma}$ is the last retained term in the sum.

By the assumption B

$$f(t) - \sum_{k=0}^{N+1} a_k t^{\alpha+k\beta} = o(t^{\alpha+(N+1)\beta}) \quad t \downarrow 0. \tag{7.16}$$

As9

Hence there exists a constant K_0 and $\delta > 0$

$$\left|f(t) - \sum_{k=0}^{N+1} a_k t^{\alpha+k\beta}\right| \leq K_0 t^{\alpha+(N+1)\beta} \quad 0 \leq t \leq \delta. \tag{7.17}$$

As10

[Actually (As10 7.17) is weaker than (As9 7.16).] Hence

$$\left|f(t) - \sum_{k=0}^{N} a_k t^{\alpha+k\beta}\right| \leq K t^{\alpha+(N+1)\beta} \quad 0 \leq t \leq \delta \tag{7.18}$$

As11

for $K = K_0 + a_{N+1}$. Thus

$$I(x) = \int_0^\delta f(t)e^{-xt}dt + E_1(x) \text{ where } E_1(x) = o(x^{-\gamma}), x \to \infty \text{ by Lemma 1} \tag{7.19}$$

$$= \int_0^\delta \sum_{k=0}^{N} a_k t^{\alpha+k\beta} e^{-xt}dt + \underbrace{\int_0^\delta \Big(f(t) - \sum_{k=0}^{N} a_k t^{\alpha+k\beta}\Big)e^{-xt}dt}_{E_2(x)} + E_1(x). \tag{7.20}$$

But by (As11 7.18)

$$\Big|\int_0^\delta f(t) - \sum_{k=0}^{N} a_k t^{\alpha+k\beta})e^{-xt}dt\Big| \le K\int_0^\delta t^{\alpha+(N+1)\beta}e^{-xt}dt \tag{7.21}$$

$$\le K\int_0^\infty t^{\alpha+(N+1)\beta}e^{-xt}dt \tag{7.22}$$

$$= Kx^{-(\alpha+(N+1)\beta+1)}\Gamma(\alpha+(N+1)\beta+1) \tag{7.23}$$

$$= \text{const. } x^{-\gamma-\beta} \tag{7.24}$$

$$= o(x^{-\gamma}) \quad x \to \infty. \tag{7.25}$$

Thus

$$I(x) = \int_0^\delta \sum_{k=0}^{N} a_k t^{\alpha+k\beta} e^{-xt}dt + E_2(x) + E_1(x) \text{ where } E_2(x) = o(x^{-\gamma}), x \to \infty \tag{7.26}$$

$$= \int_0^\infty \sum_{k=0}^{N} a_k t^{\alpha+k\beta} e^{-xt}dt + E_3(x) + E_2(x) + E_1(x) \text{ where} \tag{7.27}$$

$$E_2(x) = o(x^{-\gamma}), x \to \infty \text{ by Lemma 1} \tag{7.28}$$

$$= \sum_{k=0}^{N} a_k x^{-(\alpha+k\beta+1)}\Gamma(\alpha+k\beta+1) + E_3(x) + E_2(x) + E_1(x). \tag{7.29}$$

This proves (As8 7.15). Q.E.D.

corAS9 **Corollary 7.5** *Assume A.* $\int_{-\infty}^{\infty} |g(t)|e^{-ct^2}dt < \infty$ *for some* $c > 0$.

B. g has an asymptotic expansion

$$g(t) = \sum_{k=0}^{\infty} a_k t^k \text{ as } t \to 0. \tag{7.30}$$ As20

Let

$$I(x) = \int_{-\infty}^{\infty} g(t)e^{-xt^2}dt. \tag{7.31}$$ As21

Then

$$I(x) = \sum_{k=0}^{\infty} \frac{a_{2k}\Gamma(k+1/2)}{x^{k+1/2}} \text{ as } x \to \infty \tag{7.32}$$ As22

is an asymptotic expansion of $I(x)$.

Proof: By the same argument as in the theorem, i.e., Lemma 1, we commit an exponentially small error in replacing the integral by $\int_{-\delta}^{\delta} g(t)e^{-xt^2}dt$. We will write the remainder of the proof in an informal way because the details of justification are exactly the same as in the proof of Watson's Lemma. In fact we will write equality of series even though one must interpret all steps in terms of *finite sums* and behavior as $x \to \infty$ or $t \to 0$. In other words we will concentrate just on the algebra. Thus we have

$$E_1(x) = \text{exponentially small error} \tag{7.33}$$
$$= O(e^{-x\delta^2}). \tag{7.34}$$

$$I(x) = \int_{-\delta}^{\delta} g(t)e^{-xt^2}dt + E_1(x) \tag{7.35}$$
$$= \int_{-\delta}^{\delta} \sum_{k=0}^{\infty} a_k t^k e^{-xt^2}dt + E_1(x) \tag{7.36}$$
$$= \int_{-\delta}^{\delta} \sum_{k=0}^{\infty} a_{2k} t^{2k} e^{-xt^2}dt + E_1(x). \tag{7.37}$$

At this point we could replace the integrals by $\sum_{k=0}^{\infty} \int_{-\infty}^{\infty} t^{2k} e^{-xt^2} dt$ which can be done explicitly by integration by parts, and this would give the answer (7.32). But instead let's derive it from Watson's Lemma.

Put $s = t^2$. Then $dt = ds/2\sqrt{s}$. So

$$I(x) = 2 \int_0^{\delta} \sum_{k=0}^{\infty} a_{2k} s^k e^{-xs} \frac{ds}{2\sqrt{s}} + E_1(x) \tag{7.38}$$

$$= \int_0^{\infty} \sum_{k=0}^{\infty} a_{2k} s^{k-1/2} e^{-xs} ds + E_1(x) \tag{7.39}$$

$$= \sum_{k=0}^{\infty} \frac{a_{2k}\Gamma(k+1/2)}{x^{k+1/2}} + E(x) \text{ by Watson's Lemma with } \alpha = -\frac{1}{2},\ \beta = 1. \tag{7.40}$$

Recall that these equations are not identities but asymptotic expansions. Q.E.D.

Using the same ideas, here is another method. Let

$$f(t) = \int_{-\infty}^{\infty} e^{-xh(t)} dt$$

where h need not be quadratic but looks like this:

We assume for simplicity that $h(0) = 0$ but if not this can be subtracted from $h(t)$ changing our result by an overall factor of $e^{-th(0)}$. As in the quadratic case the main contribution to $f(t)$ will come, for large x, from a neighborhood of $t = 0$ say $|t| \leq \delta$, the error being

$$\int_{-\infty}^{\infty} e^{-tx^2} x^{2n} dx = t^{-n-1/2} \frac{(2n)!}{n! 2^{2n}} \pi^{1/2}$$

exponentially small, $e^{-xh(s)}$. (We are going to get power decay of $f(x)$.) Now since h is a minimum at $t = 0$ we have $h'(t) = 0$. Hence for small t

$$h(t) = \frac{h''(0)}{2} t^2 + O(t^3) \text{ as } t \to 0.$$

Assume $h''(0) > 0$. Our *objective* is to get the leading term in the asymptotic expansion of $f(x)$ (if there is such an expansion). So we ignore the $O(t^3)$ terms to get

$$f(x) \sim \int_{-\delta}^{\delta} e^{-x\frac{h''(0)}{2}t^2} dt \tag{7.41}$$

$$\sim \frac{\sqrt{2\pi}}{\sqrt{h''(0)x}}. \tag{7.42}$$

So

$$f(x) \sim \sqrt{\frac{2\pi}{xh''(0)}}.$$

Definition. The Γ function is

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du, \;\; x > 0. \tag{7.43}$$ As27

Note:

$$\Gamma(n+1) = n! \tag{7.44}$$ As28

Proof: $\Gamma(1) = \int_0^\infty e^{-u}du = 1 = 0!$. So (7.44) holds for $n = 0$. Induction: Note that for $x > 0$,

$$\Gamma(x+1) = \int_0^\infty u^x e^{-u} du \tag{7.45}$$ As29

$$= -u^x e^{-u}\Big|_0^\infty + \int_0^\infty e^{-u}\frac{du^x}{du}du$$

$$= \int_0^\infty e^{-u}xu^{x-1}du = x\Gamma(x).$$

So

$$\Gamma(x+1) = x\Gamma(x) \;\text{ for }\; x > 0. \tag{7.46}$$ As30

Thus if $\Gamma(n+1) = n!$ then $\Gamma(n+2) = (n+1)\Gamma(n+1) = (n+1)!$. Q.E.D.

Some values: (i) $\Gamma(1/2) = \sqrt{\pi}$.

(ii) $\Gamma(k+\frac{1}{2}) = 2^{-k}\sqrt{\pi}(2k-1)(2k-3)\cdots 3\cdot 1$, $k \geq 1$.

Proof: $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2}e^{-t}dt = \int_{-\infty}^\infty e^{-s^2}ds = \sqrt{\pi}$. The identity (ii) follows now by induction.

corAS10 **Corollary 7.6** *Suppose that $a < 0 < b$ (a or b or both could be infinite) and that $h(t) \geq 0$ for $t \in (a, b)$. Assume that $h(0) = 0$ and $h(t) > 0$ for all other t in (a, b). Assume that h is in $C^\infty(a, b)$ and that $h''(0) \neq 0$. Finally assume that*

A. $\int_a^b e^{-ch(t)} dt < \infty$ for some $c > 0$

AND

A'. $\int_{|t| \geq \delta} e^{-xh(t)} dt = o(x^{-1/2})$ as $x \to \infty$ $\forall$ $\delta > 0$.

Let

$$I(x) = \int_a^b e^{-xh(t)} dt \quad x > c.$$

Then

$$I(x) = \sqrt{\frac{2\pi}{xh''(0)}} + o(x^{-1/2}).$$

Proof: By Taylor's theorem with remainder we may write

$$h(t) = \frac{1}{2} h''(0) t^2 g(t)$$

for t near zero where g is smooth and $g(0) = 1$. Choose δ so small that $g(t) > 1/2$ for $|t| < \delta$.

Let $s = t\sqrt{g(t)}$ on $(-\delta, \delta)$. then

$$ds/dt = \sqrt{g(t)} + tg'(t)/2\sqrt{g(t)}.$$

At $t = 0$ this is one. So we may choose δ even smaller (if necessary) so that $ds/dt > 0$ on $[-\delta, \delta]$. Since s is a strictly monotone function of t on $[-\delta, \delta]$ we may solve for t in terms of s and we may expand t in terms of s to any finite order. The expansion begins:

$$t = \left(\frac{dt}{ds}\right)\Big|_{s=0} s + \frac{1}{2}\left(\frac{d^2t}{ds^2}\right)\Big|_{s=0} s^2 + \cdots.$$

But $(dt/ds)|_{s=0} = 1/(ds/dt)|_{t=0} = 1$. Hence

$$t = s + O(s^2) \text{ as } s \to 0.$$

Now

$$I(x) = \int_{-\delta}^{\delta} e^{-xh(t)} dt + \int_{a<t<b, |t| \geq \delta} e^{-xh(t)} dt.$$

By Assumption A′ the second term is $o(x^{-1/2})$. On the interval $[-\delta, \delta]$ make the change of variables $s = t\sqrt{g(t)}$. Then

$$\int_{-\delta}^{\delta} e^{-xh(t)}dt = \int_{\alpha}^{\beta} e^{-x\frac{h''(0)}{2}s^2}\frac{dt}{ds}ds \quad \alpha < 0,\ \beta > 0 \tag{7.47}$$

$$= \int_{\alpha}^{\beta} e^{-x\frac{h''(0)}{2}s^2}(1 + O(s))ds \tag{7.48}$$

$$= \frac{\Gamma(1/2)}{(xh'(0)/2)^{1/2}} + O(x^{-3/2}) \text{ as } x \to \infty \text{ by Corollary corAS9 7.5.} \tag{7.49}$$

QED

7.3.2 Stirling's formula

Corollary 7.7

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\sqrt{2\pi}e^{-x}x^{x+1/2}} = 1$$

<u>Proof</u>: Make the substitution $u = x + tx$ in Equation (As29 7.45) to find

$$\Gamma(x+1) = \int_0^{\infty} u^x e^{-u} du \tag{7.50}$$

$$= \int_{-1}^{\infty} \{x(1+t)\}^x e^{-x(1+t)} x dt \tag{7.51}$$

$$= x^{x+1}e^{-x} \int_{-1}^{\infty} e^{-xt}(1+t)^x dt \tag{7.52}$$

$$= x^{x+1}e^{-x} \int_{-1}^{\infty} e^{-x(t-\log(1+t))} dt \tag{7.53}$$

So take $h(t) = t - \log(1+t)$. Then $h'(t) = 1 - 1/(1+t)$. Therefore h has a minimum at $t = 0$. Moreover $h''(0) = 1$. Hence, by Corollary corAS10 7.6,

$$\int_{-1}^{\infty} e^{-xh(t)}dt = \sqrt{2\pi/x} + o(x^{-1/2} \quad \text{as} \quad x \to \infty.$$

This proves Stirlings formula.

The most frequently used form of Stirling's formula is for integer values of x. This takes the form:

STIRLING'S FORMULA

$$\lim_{n\to\infty} \frac{\Gamma(n+1)}{\sqrt{2\pi}e^{-n}n^{n+1/2}} = 1$$

7.4 The Method of Stationary Phase

Ref. Sec.6.5 of Bender and Orszag

Theorem 7.8 *Let f and ψ be defined on $[a,b]$ with $f \in C'[a,b]$ and ψ real with $\psi'(t) \neq 0$ on $(a,b]$ but $\psi'(a) = 0$. Assume that ψ is in $C^p[a,b]$ and $\psi^{(p)}(a) \neq 0$, while $\psi^{(n)}(a) = 0$, $n = 1, 2, \ldots, p-1$ and $f(a) \neq 0$.*

Let

$$I(x) = \int_a^b f(t) e^{ix\psi(t)} dt.$$

Then

$$I(x) = f(a) e^{ix\psi(a) \pm i\pi/2p} \Big[\frac{p!}{x \mid \psi^{(p)}(a)}\Big]^{1/p} \frac{\Gamma(1/p)}{p} O(x^{-1}) \text{ as } x \to +\infty$$

where $I = sgn\psi^{(p)}(a)$.

Example. $I(x) = \int_0^{\pi/2} e^{ix\cos t} dt$. Here $t = 0$ is a stationary point $\psi(t) = \cos t$, $f(t) = 1$, $\psi(0) = 1$, $\psi'(0) = 0$, $\psi''(0) = -1$. So $p = 2$. Therefore

$$I(x) = e^{ix - i\pi/4} \Big[\frac{2}{x}\Big]^{1/2} \cdot \frac{1}{2} \underbrace{\Gamma\Big(\frac{1}{2}\Big)}_{\sqrt{\pi}} + O(x^{-1}) \text{ as } x \to \infty.$$

<u>Proof of Theorem</u>: Write for small ε

$$I(x) = \int_a^{a+\varepsilon} f(t) e^{ix\psi(t)} dt + \int_{a+\varepsilon}^b f(t) e^{ix\psi(t)} dt.$$

The second integral is $O(x^{-1})$ as $x \to +\infty$ by the lemma. So we may ignore it for any fixed $\varepsilon > 0$. Now the idea is to choose ε so small that first integral can be adequately approximated by using the first terms of the expansion of f and ψ at a. Thus we use the approximation $f(t) \doteq f(a)$ and $\psi(t) \doteq \psi(a) + \frac{\psi^{(p)}(a)}{p!}(t-a)^p$ and we omit justification of this approximation (as do Bender and Orszag). Thus

$$I(x) = f(a) \int_a^{a+\varepsilon} e^{ix(\psi(a) + \psi^{(p)}(a)(t-a)^p/p!)} dt + O\Big(\frac{1}{x}\Big) \tag{7.54}$$

$$= f(a) e^{ix\psi(a)} \int_0^{\varepsilon} e^{ix\alpha s^p} ds + O\Big(\frac{1}{x}\Big) \tag{7.55}$$

where $\alpha = \frac{\psi^p(a)}{p!}$.

Case $\alpha > 0$. Put $s = e^{i\pi/2p}\big(\frac{u}{x\alpha}\big)^{1/p}$. Then

$$s^p = e^{i\pi/2}\frac{u}{x\alpha} = i\frac{u}{x\alpha}.$$

Therefore

$$I(x) = f(a)\Big(\frac{e^{ix\psi(a)+i\pi/2}}{(x\alpha)}\Big)\int_0^{x\alpha\varepsilon^p e^{i\pi/2p}} e^{-u}\frac{u^{\frac{1}{p}-1}}{p}du.$$

Case $\alpha < 0$ yields the other sign and C is which must be rotated down. The relevant substitution is $s = e^{-i\pi/2p}\big(\frac{u}{x|\alpha|}\big)^{1/p}$.

Case $\alpha > 0$. For $p = 2, 3, 4, \dots$ put

$$s = (e^{i\pi/2p}/(x\alpha)^{1/p})u^{1/p} \text{ on the lower half plane } \mathrm{Im}u \leq 0. \qquad (7.56) \quad \boxed{\texttt{As51}}$$

This is a well defined continuous function of u in the shaded region if one chooses the positive pth root on the positive "x" axis. Then

$$ds = \frac{e^{i\pi/2p}}{(x\alpha)^{1/p}}\ \frac{1}{p}u^{\frac{1}{p}-1}du \qquad (7.57) \quad \boxed{\texttt{As52}}$$

and also

$$s^p = (e^{i\pi/2}/x\alpha)u = iu/x\alpha. \qquad (7.58) \quad \boxed{\texttt{As53}}$$

So $u = -ix\alpha s^p$. Hence as s traverses the interval $[0, \varepsilon]$ on the positive "x" axis u traverses the contour C_1 on the negative "y" axis. Thus

$$\int_0^\varepsilon e^{ix\alpha s^p}ds = [e^{i\pi/2p}/(p(x\alpha)^{1/p})]\int_{C_1} e^{-u}u^{\frac{1}{p}-1}du \qquad (7.59) \quad \boxed{\texttt{As54}}$$

But

$$\int_{C_1} e^{-u}u^{\frac{1}{p}-1}du = \int_0^{x\alpha\varepsilon^p} e^{-u}u^{\frac{1}{p}-1}du + \int_{C_2} e^{-u}u^{\frac{1}{p}-1}du \qquad (7.60) \quad \boxed{\texttt{As55}}$$

by analyticity of $e^{-u}u^{\frac{1}{p}-1}$ in the open doted region. (The singularity at the origin is not too bad.) The first term on the right of (As55 7.60) converges to

$\Gamma(1/p)$ as $x \to \infty$ with exponentially small error. Moreover on C_2 we may write $u = re^{i\theta}$ with $r = x\alpha\varepsilon^p$ and $-\pi/2 \le \theta \le 0$. Thus

$$\Big|\int_{C_2} e^{-u} u^{1/p-1} du\Big| = \Big|\int_{-\pi/2}^{0} e^{-re^{i\theta}} r^{\frac{1}{p}-1} e^{i\theta(\frac{1-p}{p})} ir d\theta\Big| \tag{7.61}$$

$$\le r^{1/p} \int_{-\pi/2}^{0} e^{-r\cos\theta} d\theta \tag{7.62}$$

$$\le r^{1/p} \int_{0}^{\pi/2} e^{-2r/\pi} dt = r^{1/p} O\Big(\frac{1}{r}\Big) \tag{7.63}$$

$$= x^{1/p} O\Big(\frac{1}{x}\Big). \tag{7.64}$$

As57

Now combine (0), (7.59), (7.60), (??) and (7.64). Q.E.D.

7.5 The Method of Steepest Descent

Objective: To find asymptotic behavior of

$$I(x) = \int_a^b f(t) e^{x\rho(t)} dt$$

where f and ρ are *analytic*, a and b are complex or infinite, and the integral is along some contour in the complex plane.

Under these circumstances the contour can be deformed without changing the value of the integral. We take advantage of this by deforming to a contour on which the imaginary part of ρ is (essentially) constant. This will eliminate the oscillation problem.

Example 1. (From Bender and Orszag, p.281 Example 1)

$$I(x) = \int_0^1 \ln t \; e^{ixt} dt.$$

Note that the method of stationary phase fails: $\psi(t) = t$ has no stationary point. Moreover $f(t) = \ln t$ is singular at 0 so integration by parts fails.

Step 1. Deform path from to . Now along C_2, $e^{ixt} = e^{-xT} e^{ixu}$, $0 \le u \le 1$, where we have put $t = u + iT$. So for fixed $x > 0$, $\int_{C_2} \to 0$ as $T \to \infty$. Hence

$$I(x) = \int_0^{i\infty} \ln t \; e^{ixt} dt - \int_1^{1+i\infty} \ln t \; e^{ixt} dt.$$

On these paths there is no oscillation of the exponential factor because it has constant imaginary part on each path. To see this put $t = is$, $0 \le s < \infty$ on the first path and $t = 1 + is$ on the second path to get

$$I(x) = i \int_0^\infty \ln(is) e^{-xs} ds - ie^{ix} \int_0^\infty \ln(1 + is) e^{-xs} ds.$$

These are both Laplace integrals and so the problem is now in principle solved: We have gotten rid of the oscillatory integrals by deforming the path. It happens that the first integral can be done explicitly: Put $u = sx$. Then

$$i \int_0^\infty \ln(is) e^{-xs} ds = \frac{i}{x} \int_0^\infty \ln(iu/x) e^{-u} du \tag{7.65}$$

$$= (i/x)\Big[\ln(i/x) + \underbrace{\int_0^\infty (\ln u) e^{-u} du}_{-\gamma \text{ (Eulers constant)}} \Big] \tag{7.66}$$

$$= \frac{i \ln x + i\pi/2 - i\gamma}{x}. \tag{7.67}$$

The second term has an asymptotic expansion via Watson's Lemma:

$$\int_0^\infty \ln(1 + is) e^{-xs} ds = \sum_{n=1}^\infty \int_0^\infty \frac{(-is)^n}{n} e^{-xs} ds \tag{7.68}$$

$$= \sum_{n=1}^\infty \frac{(-i)^n}{n x^{n+1}} \int_0^\infty u^n e^{-u} du \tag{7.69}$$

$$= \sum_{n=1}^\infty \frac{(-i)^n (n-1)!}{x^{n+1}}. \tag{7.70}$$

Note that this is asymptotic only and not convergent because the series for $\ln(1 + is)$ only converges in a neighborhood of $s = 0$. This is all that is required for aymptotic expansion. Thus

$$I(x) \sim \frac{-i \ln n + i(\frac{\pi}{2} - \gamma)}{x} - ie^{ix} \sum_{n=1}^\infty (-i)^n \frac{(n-1)!}{x^{n+1}} \text{ as } x \to \infty.$$

7.5.1 Constancy of phase and steepness of descent

Consider an analytic function $\rho(t) = \varphi(t) + i\psi(t)$ where $t = u + iv$ is a complex variable. We have seen that the asymptotic properties of $\int_a^b f(t) e^{x\varphi(t)} dt$

depends on the peaking property of φ. It is fortuitous that these peaking properties are optimal along paths of constant ψ as we shall now see.

Let C be a curve of constant ψ and let $t \in C$.

Case a. $\rho'(t) \neq 0$. The Cauchy Riemann equations read $\varphi_u = \psi_v$, $\varphi_n = -\psi_u$. So

$$\nabla\varphi \cdot \nabla\psi = \varphi_u\psi_u + \varphi_v\psi_v = -\varphi_u\varphi_v + \varphi_v\varphi_u = 0.$$

So $\nabla\varphi$ is $\perp$ $\nabla\psi$ and therefore $\nabla\varphi$ *is parallel to* C. Hence, among all directions starting from t, φ varies most rapidly in the tangent direction of C. Thus φ ascends most rapidly in one direction along C and descends most rapidly in the other direction along C. [The level curves of ϕ are orthogonal to C.] So the curves of constant phase are also steepest curves for

$$|e^{x\rho(t)}| = e^{x\varphi(t)}.$$

In Example 1 the endpoint $t = 1$ of the constant phase path C_3 is a point with $\rho'(t) \neq 0$. Fortunately, with $t = u + iv$, $\varphi(t) \equiv Re\ it = -v$ $(= -s$ in that example) reaches a *maximum* along C_3 at the end point $t = 1$. This was the source of our success in the asymptotic expansion of $\int_{C_3}$.

Case b. $\rho'(t) = 0$. Typical behavior of ρ near such a point is illustrated by the case $\rho(t) = t^n$ near zero, $n = 2, 3, \ldots$ since the expansion of ρ around a point t_0 is $\rho(t) = a_0 + (t-t_0)^n(c+O(t-t_0))$ if $\rho^{(k)}(t_0) = 0$, $k = 1, \ldots, n-1$ and $\rho^{(n)}(t_0) \neq 0$.

Use polar coodinates: $t = re^{i\theta}$. Then

$$t^n = r^n e^{in\theta} \tag{7.71}$$
$$= r^n \cos n\theta + ir^n \sin n\theta \tag{7.72}$$
$$= \varphi + i\psi. \tag{7.73}$$

The directions of most rapid variation of φ are the directions in which $\cos n\theta = 1$ (steepest ascent) and $\cos n\theta = -1$ (steepest descent). As one increases θ one alternately reaches curves of steepest ascent and steepest descent. In either case $\sin n\theta = 0$, so these are precisely the curves of constant phase emanating from the origin.

In case $n = 2$ the graph of $\varphi(t)$ is clearly saddle shaped:

A point $t \ni \rho'(t) = 0$ is called a *saddle point* in this business. See Bender and Orszag for more pictures.

In Example 1 the maximum of φ along the curves C_1 and C_3 of constant phase occurred at the endpoint of each path where $\rho' \neq 0$. If a maximm of φ occurs at an interior point of a curve of constant phase it will necessarily be a saddle point. [This is a consequence of Cauchy-Riemann equations.]

The following example illustrates this case.

Example 2. (Bender and Orszag p.91) Asymptotic behavior of *Bessel function* $J_0(x)$.

Def. $J_0(x) = \int_{-\pi/2}^{\pi/2} \cos(x \cos\theta) d\theta/\pi$.

Step 1. Get this into standard form. Thus

$$\pi J_0(x) = Re \int_{-\pi/2}^{\pi/2} e^{ix\cos\theta} d\theta.$$

To coincide with others' notation, we also rotate the θ plane by putting $t = i\theta$. Now $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{e^t+e^{-t}}{2} = \cosh t$. so

$$\pi J_0(x) = Re \frac{1}{i} \int_{-i\frac{\pi}{2}}^{i\frac{\pi}{2}} e^{ix\cosh t} dt.$$

Step 2. Change to contour of constant phase.

Now the phase is the same at the two endspoint: $i\cosh\frac{-i\pi}{2} = i\cos\frac{-\pi}{2} = 0$ and $i\cosh\frac{i\pi}{2} = 0$ also. Nevertheless they do not actually lie on a common contour of constant phase.

To see what the constant phase contours are like write $t = u + iv$ and $\rho(t) = i\cosh(u+iv) = i[\cosh u \cosh iv + \sinh u \sinh iv] = i[\cosh u \cos v + \sinh u i \sin v]$. So $Im\rho(t) = \cosh u \cos v \equiv \psi(t)$ and $Re\rho(t) = -\sinh u \sin v \equiv \varphi(t)$. Therefore the horizontal lines $v = \pi/2$ and $v = -\pi/2$ are curves of constant phase 0. The curves $\cosh u \cos v = 1$ are also curves of constant phase 1. One of them goes through 0. Thus

$$\rho(t) = \varphi(t) + i\psi(t) \qquad \varphi(t) = -\sinh u \sin v \qquad \psi(t) = \cosh u \cos v$$

and

$$\pi J_0(x) = Re \frac{1}{i} \int_{-i\pi/2}^{i\pi/2} e^{x\rho(t)} dt \qquad t = u + iv$$

Since $Re\rho(t) = -\sinh u \sin v$, $|e^{x\rho(t)}|$ is very rapidly decreasing on the distant parts of contours C_1, C_2, C. Hence we may change from the original contour C_0 to the three contours C_1, C_2, C, of constant phase.

Step 3. Evaluation. On C_1 we have $Im\rho(t) = 0$, so $\int_{C_1} e^{x\rho(t)} dt$ is real. Hence it contributes nothing to $J_0(x)$. Similarly for $\int_{C_2}$. Consequently we have

$$\pi J_0(x) = Re \frac{1}{i} \int_C e^{ix \cosh t} dt. \tag{7.74}$$

As65

As we saw, $\varphi(t) \equiv Re\, i \cosh t = -\sinh u \sin v$ goes to $-\infty$ as we got to ∞ in either direction on C. The asymptotic behavior is therefore determined by any peaks in $\varphi(t)$ on C. These are saddle points. To find them set $\rho'(t) = 0$.

Thus $i \sinh t = 0$. $t = 0$ is clearly a saddle point. It is the only one on C. [Exercise.]

Since the asymptotic behavior is determined by a neighborhood of 0 on C we can approximate C locally by a straight line whose slope is determined thus

$$\cosh u \cos v = 1.$$

For small u and v this is $\left(1 + \frac{u^2}{2}\right)\left(1 - \frac{v^2}{2}\right) = 1$. So $1 + \frac{u^2 - v^2}{2} = 1$ to second order. Therefore

$$u^2 = v^2 : \text{ or } u = \pm v.$$

The curves of constant phase look like this.

This pattern repeats itself as one goes up the y axis $n\pi$ units. Thus if we parametrize locally on C by

$$t = (1 + i)s \qquad -\varepsilon < s < \varepsilon$$

we can expect some kind of approximation to $J_0(x)$ as $x \to \infty$. Thus, to leading order

$$\pi J_0(x) \sim Re\frac{1}{i}\int_{s=-\varepsilon}^{s=\varepsilon} e^{ix(1+t^2)}dt \tag{7.75}$$

$$\sim Re\frac{e^{ix}}{i}\int_{-\varepsilon}^{\varepsilon} e^{-2xs^2}(1+i)ds \tag{7.76}$$

$$\sim Re(1-i)e^{ix}\sqrt{\frac{\pi}{2x}} \tag{7.77}$$

$$= Re2e^{i(x-\frac{\pi}{4})}\sqrt{\frac{\pi}{nx}} \tag{7.78}$$

So

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}Ree^{i(x-\frac{\pi}{4})}$$

Therefore

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}\right) \qquad x \to \infty.$$

[Note: This agrees with the Example on p.519 if one compares $ReI(x)$.] As65

To get the full asymptotic expansion of $J_0(x)$ we must go back to (7.74) and evaluate it more carefully. On C, $\rho(t) = i + \varphi(t)$ where $\varphi(t)$ is real. We parametrize the curve by $r = -\varphi(t)$ which goes from 0 to $+\infty$ on both halves of C. Now $i\cosh t = i - r$ so $i \sinh t dt = -dr$. Therefore

$$dt = \frac{-dr}{i\sinh t} = \frac{-dr}{i\sqrt{(1+ir)^2-1}} = \frac{-dr}{i\sqrt{2r}\sqrt{+i-\frac{r}{2}}} = ????$$

$$\pi J_0(x) = Re\frac{2}{t}\int_0^\infty e^{ix}e^{-xr}\frac{1+i}{2\sqrt{r}}\ \frac{dr}{\sqrt{1+\frac{it}{2}}} = Re(1-i)e^{ix}\int_0^\infty e^{-xr}\frac{dr}{\sqrt{r}\sqrt{??}}$$

So

$$\pi J_0(x) = Re\sqrt{2}e^{i(x-\frac{\pi}{4})}\int_0^\infty \frac{1}{\sqrt{r}\sqrt{1+\frac{ir}{2}}}e^{-xr}dr.$$

To get the full asymptotic expansion one now expands $\frac{1}{\sqrt{1+\frac{ir}{2}}}$ in a power series valid for small r and informally integrates term by term getting a valid asymptotic expansion. The final form 1 according to Bender and Orszag, who get $\sqrt{1-\frac{dr}{2}}$ instead of my $\sqrt{1+\frac{ir}{2}}$, is

$$J_0(x) = \sqrt{\frac{2}{x\pi}}\left[\alpha(x)\cos\left(x-\frac{\pi}{4}\right)+\beta(x)\sin\left(x-\frac{\pi}{4}\right)\right]$$

where

$$\alpha(x) \sim \sum_{k=0}^{\infty}\frac{[\Gamma(2k+\frac{1}{2})]^2(-1)^k}{\pi(2k)!(2x)^{2k}} \qquad x\to\infty$$

and

$$\beta(x) \sim \sum_{k=0}^{\infty}\frac{[\Gamma(2k+\frac{3}{2})]^2(-1)^{k+1}}{\pi(2k+1)!(2x)^{2k+1}} \qquad x\to\infty.$$

Reference for $\alpha(x)$ and $\beta(x)$, Bender and Orszag, p.294.

7.6 Problems on asymptotic expansions

(Last homework of 615)

1. (Watson's Lemma) Suppose that $g(t)$ is a polynomially bounded function on $[0, \infty)$. That is

$$|g(t)| \leq C(1 + t^n) \qquad 0 \leq t < \infty$$

for some constant C and some integer $n \geq 0$. Suppose further that g has an asymptotic expansion as $t \downarrow 0$:

$$g(t) \sim \sum_{n=0}^{\infty} a_n t^n, \qquad t \downarrow 0.$$

Let

$$I(x) = \int_0^\infty g(t) e^{-xt} dt \qquad x > 0.$$

Prove that $I(x)$ has the asymptotic expansion

$$I(x) \sim \sum_{n=0}^{\infty} n! a_n x^{-n-1} \qquad x \to \infty.$$

2. Find the asymptotic behavior as $x \to \infty$ of $I(x)$ where

$$I(x) = \int_0^\infty [\ln(1 + it)] e^{-xt} dt.$$

3. Find the asymptotic behavior of $J_0(x)$ as $x \to \infty$ up to and including terms of order $x^{-5\frac{1}{2}}$ (i.e., $x^{-11/2}$).

8 Old Prelims

Math 615 Prelim # 1 (in class) Friday, October 10, 2003

N.B. Use a writing style which leaves no ambiguity as to what your argument is. Your writing style is a part of this test.

1. Suppose that $f : [-1, 1] \to R$ satisfies

$$\int_{-1}^{1} |f(x)|^2 dx = 21$$

and

$$\int_{-1}^{1} f(x)dx = 6.$$

What can you say about $\int_{-1}^{1} xf(x)dx$?

2.Which of the following differential operators are symmetric on the interval $[0, 1]$?

a $Lu(x) = (x^2 + 1)u''(x) + 2xu'(x) + (\sin x)u(x)$.

b. $Lu(x) = 3u''(x) + 2xu'(x) + (x^2 + 1)u(x)$.

3. Write the Green function for the boundary value problem

$$u'' - u = f \quad \text{on} \quad [0, 1]$$
$$u(0) = 0$$
$$u(1) = 0$$

in two ways.

a. in closed form (i.e. no series.)

b. as an eigenfunction expansion.

4. Which of the following are linear functionals on the given vector space V?

a. $V = R^2, \quad F((x_1, x_2)) = 3x_1 + 5x_2^2$

b. $V = C([0, 1]), \quad F(\phi) = \phi(0) + 5\int_0^1 \phi(x)x^2 dx$?

9 Appendix: Review of Real numbers, Sequences, Limits, Lim Sup, Set notation

These notes are intended to be a fast review of elementary facts concerning convergence of sequences of real numbers. The goal is to give a self contained exposition of the notion of completeness of the real number system. Completeness is responsibe for the existence of solutions to ordinary differential equations, partial differential equations, eigenvalue problems, and almost everything else that guides our intuition in analysis. The main ideas in these notes are usually taught in a freshman calculus course. Some of these ideas are developed over and over again in later courses. Chances are that this is at least the third time around for most of you ... but it may have been a while... .

Recall that a rational numbber is a number of the form m/n where m and n are integers (i.e. "whole numbers".) If $x = m/n$ is a rational number it is always possible to choose m and n to have no positive integer factors in common. We say that x is then represented in "lowest terms".

Example of an irrational number. The suare root of 2 is not rational.Proof: Assume that $x = m/n$ is represented in lowest terms and that $x^2 = 2$. Then $m^2 = 2n^2$. So m^2 is even (i.e. is a multiple of 2.) If m itself were odd then m^2 would also be odd because $(2k+1)^2 = 2(k^2+2k)+1$. So m must be even. Say $m = 2j$. Therefore $(2j)^2 = 2n^2$. So $2j^2 = n^2$. It now follows that n is also even. This contradicts our assumption that m and n have no common factors. So there is no rational number x satisfying $x^2 = 2$. Q.E.D.

Denote by $\mathbb{Q}$ the set of rational numbers and by $\mathbb{R}$ the set of real numbers. We will be using the symbols $+\infty$ and $-\infty$ in this course quite a bit. But it is a universally accepted convention not to call them real numbers. So they are not in $\mathbb{R}$. Some authors like to adjoin them to $\mathbb{R}$ and then call the result the extended real number system: $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. We will do this also. Notation: $(-\infty, \infty) = \mathbb{R}$, $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$, etc. But $\mathbb{R}$ still means the set of real numbers and excludes $\pm\infty$.

Addition, subtraction, multplication and division (but not by zero) make sense within $\mathbb{Q}$ and also within $\mathbb{R}$. (Both are fields.) Moreover $\mathbb{Q}$ and $\mathbb{R}$ both have an order relation on them. (I.e. $a < b$ has a well defined meaning, which is in fact nicely related to the algebraic operations. E.g. $a < b$ implies $a+c < b+c$, etc.) The big difference between $\mathbb{Q}$ and $\mathbb{R}$ is that $\mathbb{Q}$ is full of holes. For example the sequence of rational numbers 1, 14/10, 141/100, 1414/1000,

... “would like” to converge to $2^{1/2}$. But $2^{1/2}$ isn’t there (in $\mathbb{Q}$.) There is a hole there. The next few pages are devoted to making this anthropomorphic discussion precise.The key concept is that of a Cauchy sequence. Theorem 1 on page 4 says that there are no holes in $\mathbb{R}$.

Definition. A set $S \subset \mathbb{R}$ (real numbers) is *bounded* if there is a real number M such that $|x| \leq M$ for all x in S.

S is *bounded above* if there is a real number M such that $x \leq M$ for all x in S.

S is *bounded below* if for some M in $\mathbb{R}$, $x \geq M$ for all x in S.

Definition. If M is a number such that $x \leq M$ for all x in S then M is called an *upper bound* for S.

Definition. If L is an upper bound for S such that $L \leq M$ for all upper bounds M of S then L is called the *least upper bound* for S. We write $L = lubS$.

In order to prove anytning about the real numbers one needs , of course, a definition of them. But no axiomatic treatment of the real number system will be given here. (See Rudin, Principles of Analysis, Chapter 1, if you really want one.) Instead I am just going to list the key property - order completeness- which any definition of the real numbers must include, either in the following form or some equivalent form.

$\mathbb{R}$ is order complete: that is, if S is a nonempty set of real numbers which is bounded above then S has a least upper bound in $\mathbb{R}$.

Remark. The set $\mathbb{Q}$ of rational numbers is not order complete. For example if $S = \{x \text{ in } \mathbb{Q} : x^2 < 2\}$ then S does not have a least upper bound in $\mathbb{Q}$. But S does have a least upper bound in $\mathbb{R}$. What is it?

Definition. A *sequence* is a function from the set $Z := \{1, 2, 3, \ldots\}$ to $\mathbb{R}$.
Notation: $s(i) = s_i$.

Definition. If $\{s_n\}$ is a sequence then $\{t_k\}$ is a *subsequence* if there is a sequence integers n_k such that $n_1 < n_2 < n_3 \ldots$ and $t_k = s_{n_k}$.

Definition. A sequence $\{s_n\}$ is said to converge to a number L in $\mathbb{R}$ if for each $\varepsilon > 0$ there is an integer N such that

$$|s_n - L| < \varepsilon \text{ whenever } n \geq N.$$

Notation. $L = \lim_{n\to\infty} s_n$ or $s_n \to L$ as $n \to \infty$.

Proposition 9.1 *If s_n is an increasing sequence of real numbers bounded above then $\lim_{n\to\infty} s_n$ exists and equals lub$\{s_n : n = 1, 2, \ldots\}$.*

Proof: Let $L = \text{lub}\{s_n\}_{n=1}^{\infty}$. Then $s_n \le L$ for all n. Let $\varepsilon > 0$. Then $L - \varepsilon$ is not an upper bound for $\{s_n\}_{n=1}^{\infty}$. Thus there exists N such that $s_N > L - \varepsilon$. Since $s_n \ge s_N$ for all $n > N$ we have $s_n > L - \varepsilon$ for all $n > N$. Hence $s_n - L > -\varepsilon$. But $s_n - L \le 0$. Hence $|s_n - L| < \varepsilon$ if $n \ge N$.

Remarks. 1) If S is bounded below, then $-S$ is bounded above. By $-S$ we mean $\{-x : x \in S\}$.

2) *Greatest lower bound* is defined analogously to least upper bound.

Notation. glb S = greatest lower bound of S.

3) Notation. If S is a nonempty set of real numbers which is not bounded above we write

$$\text{lub } S = +\infty.$$

If S is nonempty and not bounded below we write glb $S = -\infty$.

Exercise. (Write out, but don't hand in) Prove: $\text{glb}(S) = -\text{lub}(-S)$ for any nonempty set S.

Definition. For any sequence $\{s_n\}_{n=1}^{\infty}$ we write $\lim_{n\to\infty} s_n = +\infty$ if for each real number M there is an integer N such that

$$s_n \ge M \text{ for all } n \ge N.$$

[Similar definition for $\lim_{n\to\infty} s_n = -\infty$.]

Now let $\{s_n\}_{n=1}^{\infty}$ be an arbitrary sequence of real numbers. Set

$$a_k = \text{lub}\{s_n : n \ge k\}.$$

Since $\{s_n : n \ge k\} \supset \{s_n : n \ge k+1\}$ we have $a_k \ge a_{k+1}$. Thus the sequence $\{a_k\}_{k=1}^{\infty}$ is a decreasing sequence.

Using the preceding proposition and definitions, we see that either

a) For all k, $a_k = +\infty$

or

b) For all k, a_k is finite (i.e., real) and the set $\{a_k\}_{k=1}^{\infty}$ is bounded below

or

c) For all k, a_k is finite and$\{a_k : k = 1, 2, \ldots\}$ is not bounded below.

In case b) $\lim_{k\to\infty} a_k$ exists (and is real) and we write

$$\lim_{n\to\infty} \sup s_n = \lim_{k\to\infty} a_k.$$

In case a) we write

$$\lim_{n\to\infty} \sup s_n = +\infty.$$

In case c) we write

$$\lim_{n\to\infty} \sup s_n = -\infty.$$

Remarks. 4) sup stands for supremum and means the same thing as lub. inf stands for infinum and means the same as glb.

5) The preceding definition of $\lim\sup s_n$ can be written succinctly in all cases as

$$\lim_{n\to\infty} \sup s_n = \lim_{k\to\infty} \sup\{s_n : n \geq k\}.$$

Similarly we define

$$\lim_{n\to\infty} \inf s_n = \lim_{k\to\infty} \inf\{s_n : n \geq k\}.$$

Proposition 9.2 *If $\lim_{n\to\infty} s_n := L$ exists and is real (i.e., finite) then*

$$\lim\sup s_n = L = \lim\inf s_n.$$

Conversely if $\lim\sup s_n = \lim\inf s_n$ and both are finite then $\lim s_n$ exists and is equal to their common value.

Proof: Assume $\lim s_n$ exists and denote it by L. Let $\varepsilon > 0$. $\exists N$ depending on ε such that $\{s_n : n \geq N\} \subset [L-\varepsilon, L+\varepsilon]$. Thus for all $k \geq N$, $a_k := \sup\{s_n : n \geq k\} \leq L+\varepsilon$. Hence $\lim_{k\to\infty} a_k \leq L+\varepsilon$. Since the inequality holds for all $\varepsilon > 0$ and the left side is independent of ε we have $\lim_{k\to\infty} a_k \leq L$. Thus $\lim_{n\to\infty} \sup s_n \leq L$. Similarly $\lim_{n\to\infty} \inf s_n \geq L$. But note that

$$\inf\{s_n : n \geq k\} \leq \sup\{s_n : n \geq k\}.$$

Hence one *always* has (for any sequence)

$$\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \sup s_n.$$

But we saw that

$$\lim_{n\to\infty} \sup s_n \le L \le \lim_{n\to\infty} \inf s_n.$$

Hence we have equality. Q.E.D.

Proof of converse. Assume

$$\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n = L$$

(L is assumed finite, i.e., real).

Let

$$a_k = \inf\{s_n : n \ge k\}$$

and

$$b_k = \sup\{s_n : n \ge k\}.$$

Then $\lim a_k = \lim b_k = L$. Let $\varepsilon > 0$. There is an integer N_1 (depending on ε, as usual) such that $|a_k - L| < \varepsilon$ whenever $k \ge N_1$. Similarly $\exists N_2 \ni |b_k - L| < \varepsilon$ whenever $k \ge N_2$. Let $N = \max\{N_1, N_2\}$. Then $|a_k - L| < \varepsilon$ and $|b_k - L| < \varepsilon$ if $k \ge N$. Thus $L - \varepsilon < a_k \le b_k < L + \varepsilon$ if $k \ge N$. In particular $L - \varepsilon < a_N \le b_N < L + \varepsilon$.

Thus

$$L - \varepsilon < a_N \le s_k \le b_N < L + \varepsilon \quad \forall\ k \ge N.$$

So

$$|s_k - L| < \varepsilon \quad \forall\ k \ge N.$$

Q.E.D.

Definition. A sequence $\{s_n\}_{n=1}^{\infty}$ is *CAUCHY* if for each $\varepsilon > 0\ \exists N \ni$

$$|s_n - s_m| < \varepsilon \quad \forall\ n, m \ge N.$$

propR3 **Proposition 9.3** *Any convergent sequence $\{s_n\}_{n=1}^{\infty}$ of real numbers is Cauchy.*

Proof: Given $\varepsilon > 0\ \exists N \ni$

$$|s_n - L| < \varepsilon/2 \quad \forall\ n \ge N.$$

Then

$$\begin{aligned} |s_n - s_m| &= |s_n - L + L - s_m| && (9.1)\\ &\le |s_n - L| + |L - s_m| && (9.2)\\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{if} \quad n, m \ge N. && (9.3) \end{aligned}$$

Q.E.D.

Theorem 9.4 *Every Cauchy sequence of real numbers converges to a real number.*

Remarks. 6) The property stated in Theorem 1 is usually referred to as the *completeness* of the set $\mathbb{R}$ of real numbers. The set of rational numbers is not complete. I.e., there is a Cauchy sequence s_n in $\mathbb{Q}$ which does not have a limit in $\mathbb{Q}$. E.g. take $s_1 = 1$, $s_2 = 1.4$, $s_3 = 1.41$, $s_4 = 1.414, \ldots$, (so that $s_n \to \sqrt{2}$, which is not in $\mathbb{Q}$).

Proof of Theorem 1: Let $\{s_n\}_{n=1}^{\infty}$ be a Cauchy sequence.

Step 1. The set $\{s_n : n = 1, 2, \ldots\}$ is bounded because if we choose $\varepsilon = 1$ in the definition of Cauchy sequence then we know that there is an integer $N \ni |s_n - s_N| < 1$ if $n \geq N$. Hence $|s_n| = |s_n - s_N + s_N| \leq |s_n - s_N| + |s_N| \leq 1 + |s_N|$ if $n \geq N$. Thus if we put $K = \max\{|s_1|, |s_2|, \ldots, |s_{N-1}|, 1 + |s_N|\}$ then $|s_n| \leq K$ for all n.

Step 2. We now know that $\limsup_{n\to\infty} s_n$ is a real number. Let $L = \limsup_{n\to\infty} s_n$.

Claim: $\lim s_n = L$. For let $\varepsilon > 0$. Choose N such that $|s_n - s_m| < \varepsilon$ $\forall\, n, m \geq N$. Then

$$s_N - \varepsilon < s_n < s_N + \varepsilon \quad \forall\, n \geq N.$$

Thus if $a_k = \sup\{s_n : n \geq k\}$ then

$$s_N - \varepsilon \leq a_k \leq s_N + \varepsilon \quad \forall\, k \geq N.$$

But by definition $L = lim_{k\to\infty} a_k$. Thus taking the limit as $k \to \infty$ we get $s_N - \varepsilon \leq L \leq s_N + \varepsilon$. So $|s_N - L| \leq \varepsilon$. Hence if $n \geq N$ then

$$|s_n - L| \leq |s_n - s_N| + |s_N - L| \leq \varepsilon + \varepsilon = 2\varepsilon \quad \forall\, n \geq N.$$

Q.E.D.

9.1 Set Theory Notation

Notation. If A and B are subsets of a set S then the *intersection* of A and B, $(A \cap B)$, is defined by

$$A \cap B = \{x \in S : x \in A \text{ and } x \in B\}.$$

The *union* of A and B is defined by

$$A \cup B = \{x \in S : x \in A \text{ or } x \in B \quad \text{or both})\}.$$

The *complement* of A is

$$A^c = \{x \in S : x \notin A\}.$$

The *difference* is

$$A - B = \{x \in A : x \notin B\}.$$

If $f : S \to T$ is a function and $B \subset T$ then $f^{-1}(B)$ is defined by

$$f^{-1}(B) = \{x \in S : f(x) \in B\}.$$

9.2 How to communicate mathematics

Before starting the following homework here are some useful tips on how to write for a reader other than yourself. The three most important rules are

Rule #1. Remove ambiguity.

Rule #2. *Remove ambiguity.*

Rule #3. Remove ambiguity.

Here are some examples of ambiguity.
Example 1. Consider the statement

$$1 = 2 \implies 3 = 4. \qquad (9.4)$$ [C1]

Is this statement true? Well, it depends on what you mean by the symbol $\implies$.

Meaning #1. "implies that". If this is what you mean by the symbol $\implies$ then the statement (9.4) says

$$1 = 2 \text{ implies that } 3 = 4. \qquad (9.5)$$ [C2]

In this case statement (9.4) is TRUE. Its equivalent to the standard if - then statement "If $1 = 2$ then $3 = 4$." All you have to do is add 2 to both sides of the equation $1 = 2$ to deduce (correctly) that $3 = 4$.

Meaning #2. "This implies that". If this is what you mean by the symbol $\implies$ then the statement (9.4)says, when translated into english,

$$1 = 2. \text{ This implies that } 3 = 4. \qquad (9.6)$$ [C3]

The statement (9.6) now consists of two sentences, one of which is false. So statement (9.6) is false and therefore statement (9.4) is FALSE.

So what to do? In principle you could define a symbol to mean anything you want (but only one meaning!). But the symbol $\implies$ is quite universally used to mean "implies that". Even the TeX command for $\implies$ is "backslashimplies", not "backslashThis implies". It would be easiest on readers to use the symbol $\implies$ always to mean "implies" or "implies that" (which has the same logical content.) Moreover there is already a standard symbol for "This imples that". Its the symbol $\therefore$.

This symbol can be translated into English in a number of logically equivalent ways. Here are some of them.

$\therefore$ can be translated as
"This implies that", "Therefore", "Hence", "So", "Consequently".

When writing for a journal you **have** to use the words, not the symbols. Moreover good prose form is better if you don't use the same one of these five phrases at the beginning of several sentences in a row. Technically you could be correct in starting five sentences in a row with the word "Therefore". But some readers might question the quality of the highschool you went to.

Often linked with the evils of the misuse of the symbol $\Longrightarrow$ is the failure to punctuate sentences. The evolution of grammar over the last 5000 years has been aimed at precision of communication. Its still far from perfect in normal social, legal and political communication. We have to do better. Keep in mind, for example, that the statement " $x = 2$." really is an english sentence. It has a subject, x, a verb, "equals", and an object, 2, AND A PERIOD at the end of the sentence. Yes, even a period can help clarity in communication of mathematics. COMMUNICATION IN MATHEMATICS CONSISTS OF A BUNCH OF GRAMMATICALLY COMPLETE SENTENCES.

If your TA can't figure out what you mean, even before he tries to figure out if you're right, expect trouble from him (and me.)

A final word. Don't think that "context" is a reasonable basis for a reader to decide how to interpret an ambiguous statement. More often than not this kind of thinking just means that you expect the reader to interpret correctly what you wrote because he/she already knows what the argument should be.

All of this was well understood quite some time ago. Here is some ancient wisdom.

" A good scientist knows how to say exactly what he means.

A good politician knows how to say exactly the opposite of what he means.

A good philosopher knows how to say things with no exact meaning."

Lezar el Gralidin, 1582 - 1631

9.3 Cute Homework Problems

1. Find the lim sup and lim inf for the following sequences:

   a) $1, 2, 3,\ 1, 2, 3,\ 1, 2, 3,\ \ldots$

   b) $\{\sin(n\pi/2)\}_{n=1}^{\infty}$

   c) $\{(1 + 1/n) \cos n\pi\}_{n=1}^{\infty}$

   d) $\{(1 + 1/n)^n\}_{n=1}^{\infty}$

2. If the lim sup of the sequence $\{s_n\}_{n=1}^{\infty}$ is equal to M, prove that the lim sup of any subsequence is $\leq M$.

3. If $\{s_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers and $\lim_{n\to\infty} \inf s_n = m$, prove that there is a subsequence of $\{s_n\}_{n=1}^{\infty}$ which converges to m. Also prove that no subsequence of $\{s_m\}_{m=1}^{\infty}$ can converge to a limit less than m.

4. Write the set of all rational numbers in $(0, 1)$ as $r_1, r_2, \ldots$. Calculate $\lim_{n\to\infty} \sup r_n$ and $\lim_{n\to\infty} \inf r_n$.

5. If $s_n \to +\infty$, prove that $\lim_{n\to\infty} \sup s_n = \infty = \lim_{n\to\infty} \inf s_n$.

6. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

7. If $f : X \to Y$ is a function, and A and B are subsets of Y, prove:

   a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

   b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

   c) $f^{-1}(A^c) = f^{-1}(A)^c$

8. Show by example that $f(A \cap B) = f(A) \cap f(B)$ is not always true.
   Show by example that $f(A^c) = f(A)^c$ is not always true.

9. Find at least seven errors in mathematical communication in the following six statements and explain whats wrong in each case.
   a. Let $f(x) = 1 - x$ for all real x.
   b. $g(x) = x - 3$
   c. The function g, which is increasing, is zero at $x = 3$

d. The function g which is increasing is greater than zero when $x > 3$.

e. The function which is increasing is less than zero when $x < 3$

f. $f(x) = g(x). \implies x = 2$

g. Rewrite item f. so that it is grammatically AND mathematically correct AND unambiguous.