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Some Results on Combinatorial Game Theory

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ABSTRACT

This thesis is divided into two parts. In the first part we discuss some theoretical results of Combinatorial Games Theory. Such discussion focuses mostly on theoretical results such as the construction of the surreal numbers and the Universal Embedding Theorem. We particularly spend some time on two different definitions of surreal numbers and the equivalence of such definitions.

In the second part we discuss and solve particular examples of combinatorial games. Concretely, using elementary ideas from Number Theory, we determine the winner of Triomineering, Tridomineering, and Lemineering, which are variations of the game of Domineering, for some boards of the form $m \times n$, for general n and $m = 2, 3, 4, 5$.

Some of the results presented in the first part are new, although not difficult to prove. Most of the results presented in the second part are new, in particular everything done in Chapter 6.

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Chapter 1

Introduction

Combinatorial Game Theory as such was invented in the 70s by John Conway [Con] (although there exist some written work before Conway's, it was he who first formalized what we know today as Combinatorial Game Theory). The games that are studied under this theory are different from others in economics. In particular, there are no chance moves such as cards or dices.

Interesting results, both theoretical and practical, have arisen throughout the years. For instance, the surreal numbers (which include real and ordinal numbers) appear as a subclass of combinatorial games, which gives another construction of the real numbers. In more practical approaches, it is common to look for better search methods to find the winner of a particular game, such as Go, Chess, and Domineering. Looking for better search methods to determine the winner of a given game can be quite a difficult problem as some games are "hardest in complexity classes beyond NP" [Fra].

A respectable amount of literature has been written on topics related to Combinatorial Game Theory and its application. By far, the classical literature on Combinatorial Games are [Con] and [BCG]. Certainly the most complete bibliography of the area is Aviezri Fraenkel's bibliography [Fra], which is constantly updated on the author's website.

Besides the classical games such as Nim and Go, new combinatorial games have also emerged through the years. For instance, the games of Triomineering, Tridomineering, and L-Tridomineering were introduced by Blanco and Fraenkel in [BlaFra]. These are zero-sum games whose rules are variations of those of the game of Domineering, presented around 1973 according to [Gar]. As it is natural to expect, some techniques that have been applied to study Domineering can also be applied to these games and further generalizations using larger tiling pieces, as pointed out in [BlaFra]. Here, we use an approach similar to that of Lachmann, Moore, and Rapaport [LMR]. In particular, we determine the winner of the games on certain boards and study the conditions that allow us to use that information to later determine the winner of larger boards.

Part I

Background

Chapter 2

Surreal Numbers

Before discussing the general theory of games, we start with the surreal numbers. Surreal numbers were invented by John Conway [Con] as a subclass of combinatorial Games. Conway's definition is inductive. Later some people have found equivalent definitions of surreal numbers without the use of inductive arguments. Here we present two definitions, which can be found in [Con] and in [Gon], respectively.

2.1 Conways's definition

Let us first look at Conway's definition. To Conway, if L and R are any two sets of numbers so that there is no $x \in L$ bigger than or equal to any element $y \in R$, then $\{L \mid R\}$ is a number. This definition depends upon what one means by "bigger than or equal to." Let us look at this definition more closely.

Definition 2.1. Let $x_1 = \{R_1 \mid L_1\}$ and $x_2 = \{R_2 \mid L_2\}$ be two surreal numbers. Then we say that x_1 is bigger than or equal to x_2, which we abbreviate by $x_1 \geq x_2$ iff there is no $x_1^R \in R_1$ so that $x_2 \geq x_1^R$ and no $x_2^L \in L_2$ is so that $x_2^L \geq x_1$.

As it is expected, whenever $x \geq y$ and $x \neq y$ we write $x < y$. Whenever $x \geq y$ does not hold, we write $x \ngeq y$ (which can either mean $y > x$ or, in the case of games, $x || y$ which will be discussed later). Also, for short, if L and R are sets so that $x \geq y$ for all $x \in L$ and all $y \in R$, then we write $L \geq R$. We also define $L < R$ and $L \ngeq R$ as before. Further, if c is a number so that for all $x \in F$ we have that $c \geq x$, then we write $c \geq F$; we write $c < F$ if $c < x$ for all $x \in F$. $F \leq c$, $F < c$ are defined accordingly.

Now we can formalize Conway's definition of (surreal) numbers

Definition 2.2. Let L and R be sets of numbers so that $L < R$. Then $\{L \mid R\}$ is a number.

We denote the collection of surreal numbers by $\mathbb{S}$. Also, to differentiate surreal numbers from the "other" numbers, they will be written in bold characters.

Note that the constructive process of surreal numbers starts with $L = R = \emptyset$. Hence, whenever we are proving statements about games using mathematical induction, it is not necessarily to check the inductive base, as any statement on $\emptyset$ is vacuously true.

The surreal numbers form a very rich algebraic structure. To see that, we now look at some of the operations that have been defined on $\mathbb{S}$.

Definition 2.3. Let $\mathbf{x} = \{\mathbf{x}^L \mid \mathbf{x}^R\}$ and $\mathbf{y}$ be surreal numbers. Then we define the following operations,

$$\mathbf{x} + \mathbf{y} = \{\mathbf{x}^L + \mathbf{y}, \mathbf{y}^L + \mathbf{x} \mid \mathbf{x}^R + \mathbf{y}, \mathbf{x} + \mathbf{y}^R\}$$

$$-\mathbf{x} = \{-\mathbf{x}^R \mid -\mathbf{x}^L\}$$

$$\mathbf{xy} = \{\mathbf{x}^L\mathbf{y} + \mathbf{x}\mathbf{y}^L - \mathbf{x}^L\mathbf{y}^L, \mathbf{x}^R\mathbf{y} + \mathbf{x}\mathbf{y}^R - \mathbf{x}^R\mathbf{y}^R \mid \mathbf{x}^L\mathbf{y} + \mathbf{x}\mathbf{y}^R - \mathbf{x}^L\mathbf{y}^R, \mathbf{x}^R\mathbf{y} + \mathbf{x}\mathbf{y}^L - \mathbf{x}^R\mathbf{y}^L\}$$

Conway [Con] showed that the surreal numbers with operations defined above define an algebraically closed Field[1].

To exemplify the definitions we have so far, we present some examples.

Example 1. If $L = R = \emptyset$ (which is vacuously a set of numbers) then we get that $\{\emptyset \mid \emptyset\}$ is a number. Without making any meaning of this yet, we assign 0 to that number

[1]We are using Conway's convention of capitalizing "field", as the domain of $\mathbb{S}$ is a proper class, and not a set. In general, we adopt the Conway's convention of capitalizing any "big" concept.

(that is $\{\emptyset \mid \emptyset\} = \mathbf{0}$).

If $L = \mathbf{0}$ and $R = \emptyset$, we have that $\{\mathbf{0} \mid \emptyset\}$ is a number, which we associate with the symbol $\mathbf{1}$.

If $L = \emptyset$ and $R = \mathbf{0}$ then $\{L \mid R\}$ is a vacuously a number, which we associate with the symbol $-\mathbf{1}$

Example 2. To exemplify the definition of "$\geq$" we now prove that $\mathbf{1} \geq \mathbf{0}$. Indeed, since $\mathbf{0} = \{\emptyset \mid \emptyset\}$ and $\mathbf{1} = \{\mathbf{0} \mid \emptyset\}$ we have that

1. No $x \in \emptyset$ is so that $\mathbf{1} \geq x$
2. No $y \in \emptyset$ is so that $y \geq \mathbf{0}$

Example 3. Let $\mathbf{x} = \{\mathbf{0} \mid \mathbf{1}\}$

It can be shown that $\mathbf{0} \geq \mathbf{0}$, and so it can also be shown that $\mathbf{0} < \mathbf{1}$. The last results are in accord to our intuition.

One of the theoretical applications of surreal numbers is that they provide a way of constructing the real numbers (or a set isomorphic to it). In other words, if $\mathbb{S}$ is the set of surreal numbers, Conway showed that $\mathbb{R} \hookrightarrow \mathbb{S}$ via f so that $f(r) = \mathbf{r}$.

Theorem 2.4. *Let $n \in \mathbb{Z}_{>0}$, then $\mathbf{n} = \{\mathbf{n} - \mathbf{1} \mid \emptyset\}$*

Proof. By induction on n. If $n = 1$, this was done in Example 1. Suppose that this is true for $n = k$. We now show that the theorem is also true for $n = k + 1$. Indeed, by definition of $+$, we have

$$\mathbf{k} + \mathbf{1} = \{(\mathbf{k} - \mathbf{1}) + \mathbf{1}, \mathbf{k} \mid\}$$

which equals $\{\mathbf{k} \mid \emptyset\}$. ■

Corollary 2.5. *Let $n \in \mathbb{Z}_{<0}$, then $-\mathbf{n} = \{\emptyset \mid -\mathbf{n} + \mathbf{1}\}$*

Proof. This is immediate from the previous theorem and the definition of the inverse of a surreal number. ■

Note that for finite n, the construction of $\mathbf{n}$ takes a finite number of steps. Furthermore, not only the integers get map to elements of $\mathbb{S}$ that can be constructed using finite steps. The *dyadic* rationals (that is, rationals of the form $\frac{j}{2^k}$ where $j, k \in \mathbb{Z}$; note that if $k \neq 0$, we may assume that $(a, 2) = 1$). By the Simplicity Rule, Theorem 3.2 (which will be discussed in the next section), we have that

$$\frac{\mathbf{2a}}{\mathbf{2^{n+1}}} = \left\{ \frac{\mathbf{2a}}{\mathbf{2^{n+1}}} \,\middle|\, \frac{\mathbf{2a+2}}{\mathbf{2^{n+1}}} \right\} = \left\{ \frac{\mathbf{a}}{\mathbf{2^{n}}} \,\middle|\, \frac{\mathbf{a+1}}{\mathbf{2^{n}}} \right\} \tag{2.1.1}$$

This last equation can be used to see that any dyadic rational $\frac{2a+1}{2^{n+1}}$ can be assigned to the surreal number $\frac{\mathbf{2a+1}}{\mathbf{2^{n+1}}}$, which by induction and Equation (2.1.1) can be constructed in a finite number of steps. In fact, it has been showed in [Con] that the dyadic rationals are the only reals that can be mapped via f to elements in $\mathbb{S}$ constructed in a finite number of steps.

To complete the embedding of $\mathbb{R}$ into $\mathbb{S}$, Conway showed the following Theorem.

Theorem 2.6 (Theorem 13, [Con])**.** *(i) Dyadic rationals are real numbers.*

(ii) Each real number has a unique expression in the form $\{L \mid R\}$, where L and R are non-empty sets of rationals such that L has no greatest, R no least, there is at most one rational in neither L nor R, and $y' < y \in L$ implies $y' \in L$, and for $z \in R$, $z < z'$ implies $z' \in R$.

(iii) Each $\{L \mid R\}$ as in (ii) represents a unique real number.

Proof. See [Con]. ■

Let $x \in \mathbb{R}$ and define L to be the set of the image under f of the set of all the dyadic rationals smaller than x, and R to be the set of the image under f of the set of all the dyadic rationals bigger than x. Then, by the previous theorem and the Simplicity Rule, we have that there exists a game $G = \{L \mid R\}$. We take this game to be the image of x under f. Clearly, we need an infinite number of steps to construct all the reals that are not dyadic rationals; in fact, Conway showed that all the other reals can be constructed in ω steps. This completes the definition of f, and the construction of an embedding of $\mathbb{R}$ into $\mathbb{S}$.

Example 4. $\mathbf{\frac{1}{3}} = \{\mathbf{\frac{1}{4}}, \mathbf{\frac{5}{16}}, \mathbf{\frac{21}{64}}, \mathbf{\frac{85}{216}}, \cdots \mid \mathbf{\frac{1}{2}}, \mathbf{\frac{3}{8}}, \mathbf{\frac{11}{32}}, \mathbf{\frac{43}{128}}, \cdots\}$

Since it has been established that there is an embedding of $\mathbb{R}$ into $\mathbb{S}$, the usual convention is to writing the image of any real number under f without bold characters. We will however conserve the distinction.

2.2 Gonshors's definition

Some people have defined surreal numbers without using inductive arguments. For instance Gonhsor [Gon] defines a surreal number as a sequence of +s and −s. Of course, this definitions turn out to be equivalent, as we will discuss here.

Definition 2.7. A *surreal number* is a function from an initial segment of the ordinals into the set $\{+, -\}$, i.e. informally, an ordinal sequence consisting of pluses and minuses which alternate. The empty sequence is included as a possibility.

For instance, $\mathbf{x}$ is the function so that $\mathbf{x}(0) = +, \mathbf{x}(1) = +$, and $\mathbf{x}(2) = -$; then $\mathbf{x}$ is the surreal number $(++-)$. Note that $\mathbf{x}$ is not defined for other ordinals. To understand the definition, recall that an *ordinal number* is the set of all its predecessors.

One of the advantages of Gonshor's definition is that surreal numbera appear as a somehow concrete objects. For instance, Conway's definition provides equivalence classes of surreal numbers whereas Gonshor's definition provides unique objects. Also, one deduces immediately from Definition 2.7 that the surreal numbers is the set $\{+,-\}^{\alpha}$.

Gonshor also defines the *length* of a surreal number $\mathbf{y}$, $\ell(\mathbf{y})$, as the least ordinal α for which $\mathbf{y}$ is undefined. For instance, if $\mathbf{x} = (++-)$, we have that $\ell(\mathbf{x}) = 3$, as $\mathbf{x}$ is defined for $0, 1, 2$ but not for 3. Note that the length of a surreal number need not to be finite; for instance, consider the sequence $(+++\cdots)$ where there are ω pluses involved.

The way Gonshor constructs a subset of $\mathbb{S}$ that is isomorphic to the real numbers is the following:

1. $()$, the empty sequence, represents 0.
2. $(\underbrace{++\ldots+}_{n \text{ times}})$ represents $n \in \mathbb{Z}_{>0}$.
3. $(\underbrace{--\ldots-}_{n \text{ times}})$ represents $-n$ where $n \in \mathbb{Z}_{>0}$.
4. If $a, b, c_1, \ldots, c_k \in \{+,-\}$ and $a \neq b$ then $(\underbrace{aa\ldots abc_1c_2,\ldots c_k}_{n \text{ times}})$ represents the rational $n|a| + \frac{|b|}{2} + \sum_{i=1}^{k} \frac{|c_i|}{2^{i+1}}$, where $|+| = 1$ and $|-| = -1$. Note that the rationals represented on this form are the dyadic rationals.

Example 5. The surreal number $(+++-+--)$ represents the rational $3-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}-\frac{1}{16} = \frac{41}{16}$

In the process of showing the equivalence between Definition 2.2 and Definition 2.7, Gonshor establishes the following order: If $\mathbf{x}$ and $\mathbf{y}$ are numbers, $\mathbf{x} < \mathbf{y}$ if $\mathbf{x}(\alpha) < \mathbf{y}(\alpha)$, where α is the first place where $\mathbf{x}$ and $\mathbf{y}$ differ, with the convention $- < 0 < +$. For instance, we have $(+-) < (+) < (++)$. This ordering is particularly easy to apply, as we only need to compare two sequences of +s and −s and look at the first place where they differ.

An important step to show the equivalence of Definition 2.2 and Definition 2.7 as well as the construction of a subset of $\mathbb{S}$ (according to Definition 2.7) isomorphic to $\mathbb{R}$ is the so-called Fundamental Existence Theorem.

Theorem 2.8. *Let L and R be two sets of surreal numbers such that $F < G$. Then there exists a unique surreal number c of minimal length such that $F < c < G$. Usually the notation $c = L \mid R$ is used.*

Proof. See [Gon]. $\blacksquare$

So given $x \in \mathbb{R}$ not a dyadic rational, one can define L and R as the sets representing all the dyadic rationals smaller than x and bigger than x, respectively. Then there exists a unique $c = L \mid R$ representing x. This c is just an infinite sequence of +s and −s, but so that such a sequence does not end in an infinite tail of the same sign. In other words, all the finite sequence of +s and −s represent dyadic rationals (which include the integers) whereas all the other infinite sequences that do not end with an infinite tail of the same signs represent the rest of $\mathbb{R}$.

Theorem 2.9. *Any $\mathbf{x} \in \mathbb{S}$ of length α can be expressed in the form $L \mid R$ where all elements of $F \cup G$ have length less than α.*

Proof. See [Gon]. $\blacksquare$

Theorem 2.10. *Suppose $L \mid R = \mathbf{c}$ and $L' \mid R' = \mathbf{d}$. Then $\mathbf{c} \leq \mathbf{d} \iff \mathbf{c} < G'$ and $F < \mathbf{d}$.*

Proof. See [Gon]

Theorems 2.8, 2.9, and 2.10 show that Definition 2.2 and Definition 2.7 are equivalent. Particularly, Theorem 2.9 tells us that every element of $\mathbb{S}$ can be expressed using elements of smaller length. This is equivalent to the inductive definition proposed in Definition 2.2.

With what we have so far, one can deduce the definition of the negative of a number. This result, which is not explicitly stated in [Gon], agrees with Definition 2.3.

Theorem 2.11. *If* $F \mid G = \mathbf{c}$*, then* $-G \mid -F = -\mathbf{c}$.

Proof. Since $F \mid G = \mathbf{c}$, then $F < \mathbf{c} < G$, and so $-G < -\mathbf{c} < -F$. Now, the minimality of $\ell(\mathbf{c})$ guarantees the minimality of $\ell(-\mathbf{c})$, and so $-G \mid -F = -\mathbf{c}$. ∎

The way Gonshor defines operations with surreal numbers is the same as Conway's, which we have already discussed.

Chapter 3

Combinatorial Games

The games we work with in this paper belong to a particular kind called *combinatorial games*. Different authors define combinatorial games in slightly different ways.

3.1 Definition and Basic Results

The most common definition is the following,

Definition 3.1. A *combinatorial game* is any game satisfying

1. There are only two players (usually called Left and Right).
2. It is deterministic, that is, there are no random moves.
3. It is finite, that is, every play of the game ends after a finite number of moves.
4. It is a game with perfect information, that is, the result of every move by one player is known to the other player.
5. It is a zero-sum game (games that end with a win for one player and a loss for the other player, or a draw for both players).

The above definition is sometimes called *finite acyclic combinatorial games*.

Interesting results arise if one relaxes the above definition. For instance, if we do not require our games to be finite and be zero-sum, we could include chess and other games where one position can be reached multiple times. This kind of games are called *loopy games*. We are not going to deal with this kind of games.

Several examples of combinatorial games can be found in [BCG]. Hereinafter, whenever we use the word "game," we shall mean "combinatorial game."

The formal definition of a game, as presented in [Con] is a recursive one. For a game G, we write $G = \{G^L \mid G^R\}$ where G^L and G^R are themselves games. They represent the moves available to Left and Right, respectively. Recall that if G^L is $\geq$ than any member of G^R, we say that the game G is a *number* according to [Con]. The set of games have a rich structure that is presented in detail in [Con].

The following classification is made: the value of a game G is

1. Negative if Horizontal is able to win. We express this by $G < 0$.

2. Positive if Vertical is able to win. We express this by $G > 0$.

3. Zero if the second player is able to win. We express this by $G = 0$.

4. Fuzzy if the first player is able to win. We express this by $G||0$.

By condition (5) of the definition, it is clear that any combinatorial game lies in one of these four categories.

The previous classification is based on who wins a combinatorial game. There is another important classification that is made which is based on the kind of moves the players are allowed to use. Concretely, if in a game G both players are allowed to used the same moves, then G is called *impartial.* On the other hand, if Right and Left have different set of moves when playing game H, then H is called *partizan.*

One of the classical examples of games is Nim, where each player subtracts from a pile of coins a certain number of them. The winner is the one that removes the last coin. The game of Nim is impartial. One of the most important results in Combinatorial

Game Theory is that all *short* (i.e., finite) impartial games are equivalent to the game of Nim played with a pile with n coins, for some non-negative integer n. According to [Con], this result was found independently by Sprague and Grundy.

Theorem 3.2 (Simplicity Rule. Theorem 11, [Con])**.** *Let* $\mathbf{x} = \{\mathbf{x}^L \mid \mathbf{x}^L\}$ *and suppose that some number* $\mathbf{z} = \{\mathbf{z}^L \mid \mathbf{z}^R\}$ *satisfies* $\mathbf{x}^L < \mathbf{z} < \mathbf{x}^R$*, but no number from* $\mathbf{z}^L$ *or* $\mathbf{z}^R$ *satisfies that condition. Then* $\mathbf{z} = \mathbf{x}$*.*

Proof. See [Con]. ■

Definition 3.3. Let G be a game and suppose that two different Left options G^{L_1} and G^{L_2} are so that $G^{L_1} \geq G^{L_2}$. Then we say that G^{L_2} is *dominated* by G^{L_1}. The same happens if we have two Right options that can be compared.

Theorem 3.4. *Deleting any dominated option does not alter the value of a game* G*.*

Proof. See [Con]. ■

This is one of the reasons why Conway's definition of combinatorial games produce equivalence classes rather than unique objects. For instance, $\{\mathbf{1} \mid \mathbf{0}\}, \{\mathbf{0}, \mathbf{1} \mid \mathbf{0}\}$, and $\{-\mathbf{1}, \mathbf{0}, \mathbf{1} \mid \mathbf{0}\}$ are all the same game, by the previous theorem.

The Simplicity Rule and Theorem 3.4 will allow us to express games of the form

$$G = \{\mathbf{number}_1, \ldots, \mathbf{number}_n \mid \mathbf{number}'_1, \ldots, \mathbf{number}'_m\}$$

for finite n and m. Since any two numbers can be compared, games of the previous form can be reduced to

$$G = \{\mathbf{Number}_1 \mid \mathbf{Number}_2\}$$

We note that a game G of the previous form can be written as the expression $(\{+,-\}^\alpha \mid \{+,-\}^\alpha)$, where A^α denotes all the strings, finite or infinite, formed with elements of A.

3.2 Switches

A *switch* is a game of the form $\{\mathbf{x} \mid \mathbf{y}\}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{S}$ and $\mathbf{y} \geq \mathbf{x}$.

Theorem 3.5. *Let $C = (\{+,-\}^{\alpha} \mid \{+,-\}^{\alpha})$ and A be the class of switches. Then there is a one-to-one correspondence between C and A.*

Proof. Let $x \in C$, then x is of the form $\mathbf{x}_1 \mid \mathbf{x}_2$ by definition, where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}$. The converse is also easily verified, if $G = \{\mathbf{x_1} \mid \mathbf{x_2}\}$, where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}$ then there exist a sequence of $+$s and $-$s that represent $\mathbf{x}_1, \mathbf{x}_2$. ■

The inductive construction of games due to Conway and Theorem 3.5 gives us a new description of games. Namely any combinatorial game is a sequence of elements of C separated by commas and balanced parenthesis. For instance, $\{\mathbf{0} \mid \mathbf{0}\} = \mathbf{0}$ can be written as $[|]$, as $\mathbf{0}$ is represented by the empty string. Also, note that $= \{\mathbf{1} \mid \mathbf{0}, *\}$ can be represented by $[\,+ \mid\ [], [|]\,]$. Note however that with this definition of games we still can have different representations for the same game, and so, unlike with surreal numbers and switches, we do not obtain unique objects.

Both [Con] and [Gon] have given ways of ordering the surreal numbers. Whichever method one uses, the following theorem gives us a way of ordering switches.

Theorem 3.6. *Let $\{\mathbf{x} \mid \mathbf{y}\}$ and $\{\mathbf{x}' \mid \mathbf{y}'\}$ be two switch games, then*

1. $\{\mathbf{x} \mid \mathbf{y}\} \geq \{\mathbf{x}' \mid \mathbf{y}'\}$ *iff* $\mathbf{x} \geq \mathbf{x}'$ *and* $\mathbf{y} \geq \mathbf{y}'$.
2. *If it is not the case that $\mathbf{x} \geq \mathbf{x}'$ and $\mathbf{y} \geq \mathbf{y}'$, then $\{\mathbf{x} \mid \mathbf{y}\}$ and $\{\mathbf{x}' \mid \mathbf{y}'\}$ cannot be compared.*

Proof. Note that

$$
\begin{aligned}
& \{\mathbf{x} \mid \mathbf{y}\} \geq \{\mathbf{x}' \mid \mathbf{y}'\} \\
\iff & \{\mathbf{x} \mid \mathbf{y}\} + \{-\mathbf{y}' \mid -\mathbf{x}'\} \geq \mathbf{0} \\
\iff & \{\mathbf{x} + \{-\mathbf{y}' \mid -\mathbf{x}'\}, -\mathbf{y}' + \{\mathbf{x} \mid \mathbf{y}\} \mid \mathbf{y} + \{-\mathbf{y}' \mid -\mathbf{x}'\}, -\mathbf{x}' + \{\mathbf{x} \mid \mathbf{y}\}\} \geq \mathbf{0} \\
\iff & \{\{\mathbf{x} - \mathbf{y}' \mid \mathbf{x} - \mathbf{x}'\}, \{\mathbf{x} - \mathbf{y}' \mid \mathbf{y} - \mathbf{y}'\} \mid \{\mathbf{y} - \mathbf{y}' \mid \mathbf{y} - \mathbf{x}'\}, \{\mathbf{x} - \mathbf{x}' \mid \mathbf{y} - \mathbf{x}'\}\} \geq \mathbf{0}
\end{aligned}
$$

which is true, unless the last game has a right option G^R so that $\mathbf{0} \geq G^R$.

Suppose that $\mathbf{x}' \geq \mathbf{x}$ and $\mathbf{y}' \geq \mathbf{y}$ $\mathbf{0} \geq \{\mathbf{y} - \mathbf{y}' \mid \mathbf{y} - \mathbf{x}'\}$. We recognize two cases,

1. $\mathbf{x}' \geq \mathbf{y}'$. In this case, $\mathbf{0} \geq \mathbf{y} - \mathbf{x}'$, and so $\{\{\mathbf{y} - \mathbf{y}'\} \mid \{\mathbf{y} - \mathbf{x}'\}\}$ is fuzzy.
2. $\mathbf{y}' > \mathbf{x}'$. In this case, $\{\mathbf{y} - \mathbf{y}' \mid \mathbf{y} - \mathbf{x}'\}$ is positive.

A similar argument for $\{\mathbf{x} - \mathbf{x}' \mid \mathbf{y} - \mathbf{x}'\}$ shows that

$$\{\{\mathbf{x} - \mathbf{y}' \mid \mathbf{x} - \mathbf{x}'\}, \{\mathbf{x} - \mathbf{y}' \mid \mathbf{y} - \mathbf{y}'\} \mid \{\mathbf{y} - \mathbf{y}' \mid \mathbf{y} - \mathbf{x}'\}, \{\mathbf{x} - \mathbf{x}' \mid \mathbf{y} - \mathbf{x}'\}\}$$

has no right option G^R so that $\mathbf{0} \geq G^R$. This implies that

$$\{\{\mathbf{x} - \mathbf{y}' \mid \mathbf{x} - \mathbf{x}'\}, \{\mathbf{x} - \mathbf{y}' \mid \mathbf{y} - \mathbf{y}'\} \mid \{\mathbf{y} - \mathbf{y}' \mid \mathbf{y} - \mathbf{x}'\}, \{\mathbf{x} - \mathbf{x}' \mid \mathbf{y} - \mathbf{x}'\}\} \geq \mathbf{0}$$

Suppose that

$$\{\{\mathbf{x} - \mathbf{y}' \mid \mathbf{x} - \mathbf{x}'\}, \{\mathbf{x} - \mathbf{y}' \mid \mathbf{y} - \mathbf{y}'\} \mid \{\mathbf{y} - \mathbf{y}' \mid -\mathbf{x}'\}, \{\mathbf{x} - \mathbf{x}' \mid \mathbf{y} - \mathbf{x}'\}\} \geq \mathbf{0}$$

This implies that it is *not* true that

1. $\mathbf{0} \geq \{\mathbf{y} - \mathbf{y}' \mid \mathbf{y} - \mathbf{x}'\}$. This impossibility is reached when $\mathbf{y} - \mathbf{y}' \geq \mathbf{0}$ or $\mathbf{y} \geq \mathbf{y}'$.
2. $\mathbf{0} \geq \{\mathbf{x} - \mathbf{x}' \mid \mathbf{y} - \mathbf{x}'\}$. This impossibility is reached when $\mathbf{x} - \mathbf{x}' \geq \mathbf{0}$ or $\mathbf{x} \geq \mathbf{x}'$.

For (2), consider that $\{\mathbf{6} \mid \mathbf{1}\} \mid\mid \{\mathbf{5} \mid \mathbf{2}\}$. ■

Theorem 3.7. *Let $\mathfrak{G}$ be the class of games. Then $\mathfrak{G}$ forms a Vector Space over $\mathbb{S}$.*

Proof. From the definition of the sum and the inverse of a game, one readily sees that $\mathfrak{G}$ forms an abelian group. Now, from the definition of multiplication of games by numbers, we also know that this operation is closed. We still need to verify two conditions: (1) That for any two surreal numbers $\mathbf{x}, \mathbf{y}$ and any game G, $(\mathbf{x}+\mathbf{y})G = \mathbf{x}\,G + \mathbf{y}\,G$ and (2) $\mathbf{x}(\mathbf{y}\,G) = (\mathbf{x}\,\mathbf{y})G$. This can be verified directly using an inductive argument as follows: let $\mathbf{x} = \{\mathbf{x}^L \mid \mathbf{x}^R\}$ and $\mathbf{y} = \{\mathbf{y}^L \mid \mathbf{y}^R\}$ be two surreal numbers and $G = \{G^L \mid G^R\}$ be a game. Then,

$$
\begin{aligned}
(\mathbf{x}+\mathbf{y})G = \{&(\mathbf{x}+\mathbf{y})^L G + (\mathbf{x}+\mathbf{y})G^L - (\mathbf{x}+\mathbf{y})^L G^L, (\mathbf{x}+\mathbf{y})^R G + (\mathbf{x}+\mathbf{y})G^R - (\mathbf{x}+\mathbf{y})^R G^R \mid \\
&(\mathbf{x}+\mathbf{y})^L G + (\mathbf{x}+\mathbf{y})G^R - (\mathbf{x}+\mathbf{y})^L G^R, (\mathbf{x}+\mathbf{y})^R G + (\mathbf{x}+\mathbf{y})G^L - (\mathbf{x}+\mathbf{y})^R G^L\} \\
= \{&(\mathbf{x}^L+\mathbf{y}, \mathbf{x}+\mathbf{y}^L)G + (\mathbf{x}+\mathbf{y})G^L - (\mathbf{x}^L+\mathbf{y}, \mathbf{x}+\mathbf{y}^L)G^L, \\
&(\mathbf{x}^R+\mathbf{y}, \mathbf{x}+\mathbf{y}^R)G + (\mathbf{x}+\mathbf{y})G^R - (\mathbf{x}^R+\mathbf{y}, \mathbf{x}+\mathbf{y}^R)G^R \mid \\
&(\mathbf{x}^L+\mathbf{y}, \mathbf{x}+\mathbf{y}^L)G + (\mathbf{x}+\mathbf{y})G^R - (\mathbf{x}^L+\mathbf{y}, \mathbf{x}+\mathbf{y}^L)G^R, \\
&(\mathbf{x}^R+\mathbf{y}, \mathbf{x}+\mathbf{y}^R)G + (\mathbf{x}+\mathbf{y})G^L - (\mathbf{x}^R+\mathbf{y}, \mathbf{x}+\mathbf{y}^R)G^L\} \\
= \{&(\mathbf{x}^L+\mathbf{y})G + (\mathbf{x}+\mathbf{y})G^L - (\mathbf{x}^L+\mathbf{y})G^L, (\mathbf{x}+\mathbf{y}^L)G + (\mathbf{x}+\mathbf{y})G^L - (\mathbf{x}+\mathbf{y}^L)G^L, \\
&(\mathbf{x}^R+\mathbf{y})G + (\mathbf{x}+\mathbf{y})G^R - (\mathbf{x}^R+\mathbf{y})G^R, (\mathbf{x}+\mathbf{y}^R)G + (\mathbf{x}+\mathbf{y})G^R - (\mathbf{x}+\mathbf{y}^R)G^R \mid \\
&(\mathbf{x}^L+\mathbf{y})G + (\mathbf{x}+\mathbf{y})G^R - (\mathbf{x}^L+\mathbf{y})G^R, (\mathbf{x}+\mathbf{y}^L)G + (\mathbf{x}+\mathbf{y})G^R - (\mathbf{x}+\mathbf{y}^L)G^R, \\
&(\mathbf{x}^R+\mathbf{y})G + (\mathbf{x}+\mathbf{y})G^L - (\mathbf{x}^R+\mathbf{y})G^L, (\mathbf{x}+\mathbf{y}^R)G + (\mathbf{x}+\mathbf{y})G^L - (\mathbf{x}+\mathbf{y}^R)G^L\} \\
= \{&(\mathbf{x}^L G + \mathbf{x} G^L - \mathbf{x}^L G^L) + \mathbf{y} G, \mathbf{x} G + \mathbf{x} G + (\mathbf{y}^L G + \mathbf{y} G^L - \mathbf{y}^L G^L), \\
&(\mathbf{x}^R G + \mathbf{x} G^R - \mathbf{x}^R G^R) + \mathbf{y} G, \mathbf{x} G + \mathbf{x} G + (\mathbf{y}^R G + \mathbf{y} G^R - \mathbf{y}^R G^R) \mid \\
&(\mathbf{x}^R G + \mathbf{x} G^R - \mathbf{x}^R G^R) + \mathbf{y} G, \mathbf{x} G + \mathbf{x} G + (\mathbf{y}^R G + \mathbf{y} G^R - \mathbf{y}^L G^R), \\
&(\mathbf{x}^L G + \mathbf{x} G^L - \mathbf{x}^R G^L) + \mathbf{y} G, \mathbf{x} G + \mathbf{x} G + (\mathbf{y}^R G + \mathbf{y} G^L - \mathbf{y}^R G^L)\} \\
= \mathbf{x}&\, G + \mathbf{y}\, G
\end{aligned}
$$

Similarly, one can show that $\mathbf{x}(\mathbf{y}\, G) = (\mathbf{x}\,\mathbf{y})G$. This completes the proof. ■

3.3 A Conjecture of Conway

Conway made a conjecture regarding the structure of the class of games. This conjecture was proven by Jacob Lurie [Lur] that the class of combinatorial games constitutes a "universally embedding" partially ordered abelian group. Formally,

Theorem 3.8. *Let $\mathfrak{G}$ represent the group of games and let $S \subset S'$ be partially ordered abelian groups. Suppose that $\phi : S \longrightarrow \mathfrak{G}$ is an order-preserving homomorphism. Then there exists an order-preserving homomorphism $\phi' : S' \longrightarrow \mathfrak{G}$ such that $\phi'_{|S} = \phi$*

Proof. See [Lur]. ■

To exemplify the previous theorem, consider the partially ordered abelian group H formed by all subsets of $[n] = \{1, \ldots, n\}$ where $n \in \mathbb{Z}_{>0}$, with group operation $\cup$ and partial order $\subset$. That is, for $A, B \in 2^{[n]}$ (the power set of $[n]$) we have that $A \geq B \iff A \subset B$.

Note that $|H| = 2^n$. Also, note that $N = p_1, \cdots, p_n$ where all the p_is are different primes, is a positive integer with 2^n divisors. Let $H' \subset \mathfrak{G}$ be a partially ordered abelian group formed by all positive divisors of N, with partial order " $|$ "; that is, $a \geq b \iff a \mid b$. By looking at the lattice of H and H' one readily sees that there is an order-preserving isomorphism ϕ between them. Now, if we fixed n and define similar partially ordered abelian groups H_1 and H'_1 for $m > n$, we would see that it is possible to define $\phi' : H_1 \longrightarrow H'_1$ so that $\phi'_{|H} = \phi$.

Part II

New Combinatorial Games

Chapter 4

The Games

The games we consider here are variations of the game of Domineering, invented by Göran Anderson in 1973. The Game of Domineering consists of two players, usually called Horizontal and Vertical that tile a board (not-necessarily rectangular) alternately with dominoes according to their names. That is, Vertical tiles the board with dominoes placed vertically, and Horizontal tiles the board with dominoes placed horizontally.

Blanco and Fraenkel [BlaFra] created and studied variations of the game of Domineering. These are described below.

Triomineering In this game we substitute the domino by a "straight" triomino; that is, a 3×1 tile. The rules of this game are exactly the same as for Domineering: the two players, Vertical and Horizontal, tile alternately vertically and horizontally, respectively. Overlapping is not permitted. The player making the last move wins.

Tridomineering In this game the players are allowed to use both "straight" triominoes and a dominoes. The rules of this game are exactly the same as for Domineering.

L-Tridomineering In this game the players are allowed to use dominoes and triominoes, that also include L-shaped triominoes; that is, 2×2 boards with a corner removed. The other rules are the same.

Lemineering In this game the players use only the L-shaped triomino. The other rules are the same. Note that this is an impartial game.

The following tables present some values when the games are played on small rectangular boards. Some of the values were considerably difficult to read, and so we used

F to denote a fuzzy game rather than writing down the actual value. Similarly, an H represents a win for Horizontal.

Table 4.1: Values of Triomineering for $m \times n$ boards.

m/n	1	2	3	4	5	6
1	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-2}$
2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{-4}$
3	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{\pm 2}$	$\mathbf{\{3 \mid -3/2\}}$	$\mathbf{\{4 \mid -1, -1*\}}$	$\mathbf{\{4 \mid 0 \mid\mid\mid -1/2 \mid\mid -1 \mid -2\}}$
4	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{\{3/2 \mid -3\}}$	$\mathbf{\pm 5/2}$	$\mathbf{\{3 \mid -2, -2*\}}$	$\mathbf{\{3 \mid -3/2 \mid\mid\mid -7/4 \mid\mid -3 \mid -4\}}$
5	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{\{1, 1* \mid -4\}}$	$\mathbf{\{2, 2* \mid -3\}}$	$\mathbf{\pm 2}$	$\mathbf{\{-3 \mid\mid -3, -3* \mid -8\}}$
6	$\mathbf{2}$	$\mathbf{4}$	$\mathbf{\{2 \mid 1 \mid\mid 1/2 \mid\mid\mid 0 \mid -4\}}$	$\mathbf{\{4 \mid 3 \mid\mid 7/4 \mid\mid\mid 3/2 \mid -3\}}$	$\mathbf{\{8 \mid 3, 3* \mid\mid 3\}}$	F

Table 4.2: Values of Tridomineering for $m \times n$ boards.

m/n	1	2	3	4	5
1	$\mathbf{0}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-2}$	$\mathbf{-2}$
2	$\mathbf{1}$	$\mathbf{\pm 1}$	$\mathbf{\{2 \mid -1\}}$	$\mathbf{\{2 \mid 0 \mid\mid -1*\}}$	F
3	$\mathbf{1}$	$\mathbf{\{1 \mid -2\}}$	$\mathbf{\pm 2}$	$\mathbf{\{2 \mid -1 \mid\mid -2\uparrow\}}$	H
4	$\mathbf{2}$	$\mathbf{\{1* \mid\mid 0 \mid -2\}}$	$\mathbf{\{2\downarrow \mid\mid 1 \mid -2\}}$	$\mathbf{\pm 1*}$	F
5	$\mathbf{2}$	$\mathbf{\{1, \{2 \mid 0\} \mid 0, \{0 \mid -3\}\}}$	$\mathbf{\{7/4, \{4 \mid 1\} \mid 1, \{1 \mid -3\}\}}$	F	F

The following definitions also appeared in [BlaFra]. We use them in the next section.

Definition 4.1. Two boards F and G placed next to each other are said to be *concatenated horizontally* or simply *concatenated*, if one can place a horizontal domino so that it covers one square from F and one from G.

We denote any concatenation of G and F by $GF_{(i,j)}$, where (i, j) denotes the smallest positive integers i, j such that row i of G is aligned to row j of F (assuming that we

Table 4.3: Values of L-Tridomineering for small rectangular boards.

	1	2	3	4	5
1	$\mathbf{0}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-2}$	$\mathbf{-2}$
2	$\mathbf{1}$	$\mathbf{\pm 1}$	$\{\mathbf{2}\vert\mathbf{-1}\}$	$\{\mathbf{1}*, \{\mathbf{2}\vert\mathbf{0}\}\vert\mathbf{-1}*\}$	$\{\mathbf{0}, \{\mathbf{3}\vert\mathbf{0}\}\vert\mathbf{0}, \{\mathbf{0}\vert\mathbf{-2}\}\}$
3	$\mathbf{1}$	$\{\mathbf{1}\vert\mathbf{-2}\}$	$\mathbf{\pm 2}$	H	H
4	$\mathbf{2}$	$\{\mathbf{1}*\vert\{\mathbf{0}\vert\mathbf{-2}\}, \mathbf{-1}*\}$	V	F	F
5	$\mathbf{2}$	$\{\mathbf{0}, \{\mathbf{2}\vert\mathbf{0}\}\vert\mathbf{0}, \{\mathbf{0}\vert\mathbf{-3}\}\}$	V	F	F

enumerate starting from the upper row, as is common when labeling the rows of a matrix). Whenever the pair (i, j) is not important for a result to hold, we simply denote a concatenation of G and F by GF, as it is the case in Theorem 5.3.

Definition 4.2. Let G and F be subsets of rectangular boards, and $GF_{(i,j)}$ be a concatenation of them. If each column of $GF_{(i,j)}$ intersects only one of the two boards, then G is *i-smoothly left-aligned* to F, or simply (i, j)-*left-aligned*.

If G is (i, j)-left-aligned to F for all possible (i, j), then we simply say that G is left-aligned to F.

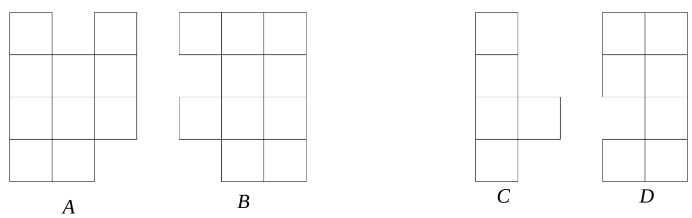

Figure 4.1: Aligned and non-aligned boards.

Of course, Definition 4.2 depends on the way we concatenate boards. For instance, Figure 4 depicts an example where A is $(1, 1)$-aligned to B but not $(4, 1)$-aligned, as $AB_{(1,4)}$ will contain in its third column squares from both boards. Also, C is not $(1, 1)$-aligned to D, whereas D is $(1, 1)$-aligned to C, as each column of $DC_{(1,1)}$ intersects only one of the two boards; in fact, D is left-aligned to C.

Chapter 5

Ideas From Number Theory

Definition 5.1. Denote by $\mathbb{Z}_{\geq n}$ the set of all positive integers greater or equal to n. Let A be a subset of the positive integers. We say that A is a *non-negative basis* for $\mathbb{Z}_{\geq n}$ if any integer greater or equal to n can be written as a non-negative integer (linear) combination of elements of A.

To illustrate the above definition, consider the set $A = \{4, 5, 6\}$. One can easily see that A is a non-negative basis for $\mathbb{Z}_{\geq 8}$. Indeed, any non-negative integer combination of elements of A looks like $4k_1 + 5k_2 + 6k_3$ for $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$ not all of them zero. So considering the positive integers modulo 4 we get that

- For n of the form $4k, k > 0$, we use $k_2 = k_3 = 0$.
- For n of the form $4k + 1, k > 0$, we use $k_1 = k - 1,\ k_2 = 1,\ k_3 = 0$.
- For n of the form $4k + 2, k > 0$, we use $k_1 = k - 1,\ k_2 = 0,\ k_3 = 1$.
- For n of the form $4k + 3, k > 1$, we use $k_1 = k - 2,\ k_2 = 1,\ k_3 = 1$.

The following is a result on non-negative basis,

Theorem 5.2. *Let $k \in \mathbb{Z}_{>0}$, then the set $A = \{k, k+1, \ldots, 2k\}$ forms a non-negative basis for $\mathbb{Z}_{\geq k}$*

Proof. First note that A is a complete system of remainders modulo k. Clearly, one can express all the integers between k and $2k$ using elements from A. To see how to represent $n > 2k$, we use the division algorithm to express n in the form $qk + r$ where $0 \leq r < k$. So $r \in A \cup \{0\}$, which clearly implies that n can be expressed as an integer linear combination of elements of A. ■

For instance, the set $\{1, 2\}$ forms a non-negative basis for $\mathbb{Z}_{\geq 1} = \mathbb{Z}_{>0}$, as one can readily check directly.

A non-negative basis can be used to determine the winner of certain games. This application relies on the following theorem, which is a generalization of a result of [LMR]. This theorem applies to Triomineering and Tridomineering.

Theorem 5.3. *Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a set of k left-aligned boards (i.e., each pair of boards of $\mathcal{F}$ is left-aligned) so that Horizontal can win on every one of them. If G is constructed by concatenating any number of –not necessarily distinct copies of– boards from $\mathcal{F}$, then $G < 0$.*

Proof. The idea is to notice that Horizontal has the power to divide G into smaller boards by restricting himself from tiling pieces from more than one board. The following is a winning strategy for Horizontal: Divide G into pieces and play exclusively on a given copy of a game, without ever tiling more than one copy at a time. Vertical is unable to play on more than one copy by left-alignment. In this way, Horizontal establishes vertical boundaries, which cannot be crossed by Vertical. Since $F_i < 0$, for $i = 1, 2, \ldots, k$, Horizontal can force a win in each copy that forms G. In this fashion, G becomes the sum of the copies of $\mathcal{F}$, and so Horizontal can win on G. ■

When playing on a rectangular board with 2 or three rows, each player can easily break the game into the sum of smaller games. Vertical can do that simply by placing a tile, and Horizontal can do that by restricting the moves he performs. We shall see that for certain boards it is possible to find a non-negative basis, which gives Horizontal the upper hand.

Note that this technique cannot be directly applied to L-Tridomineering, as each player is allowed to use an L-shaped triomino, and so Horizontal no longer has the advantage of restricting the play to a given copy. However, the data collected so far lead to the belief that, with some variations, Theorem 5.3 still holds.

Chapter 6

Results On The New Games

We are going to focus on games of the form $m \times n$, where $m = 2, 3, 4, 5$. The theoretical values of the games were computed using a plug-in for Siegel's *Combinatorial Game Suite* (CGS) [Sie]. Such a plug-in was needed in order for CGS to understand the rules of the variations of Domineering. No change was done to code of CGS. For the computer experiments, we used a Dell PowerEdge 1750 server with dual 3 GHz Xeon processors and 4 GB of memory running Linux 2.4.21 and a Sun Enterprise 420R server with quad UltraSPARCs processors and 4GB of memory running Solaris 8.

As a matter of notation, we use in the tables, we denote a win for Horizontal by H, a win for Vertical by V, and a win for the first player with an F.

The technique applied is the following: Given m, we try to find n for which the game played on the $m \times n$ board is negative. Then by Theorem 5.3, concatenating any number of those boards will produce a negative game. If we can find a set of positive integers that are positive generators for $\mathbb{Z}_{\geq k}$ for some $k \in \mathbb{Z}_{>0}$, then we can determine the winner for a general game played on an $m \times n$ board, where $n \geq k$.

For the sake of completeness, we mention that Horizontal wins Triomineering on $2 \times n$ boards for all $n \geq 2$, as Vertical cannot move at all on any such a board. For $n = 1$, the first player to move looses.

It is important to notice that any result obtained for boards of the form $m \times n$, $m = 2, 3, 4$ also holds true for boards of the form $n \times m$, with the respective changes, i.e., if Horizontal wins on a given board of the form $m \times n$, then Vertical wins on the $n \times m$

board. We call the board of the form $n \times m$ the *negative* of the $m \times n$ board. If a board is a win for the first or second player, its negative is also a win for the first or second player, respectively.

6.1 Results on Triomineering

6.1.1 Triomineering for $3 \times n$ boards

All the values found so far are fuzzy. More information is needed to determine the winner for an arbitrary n. However, by Theorem 5.3, we can conclude that Horizontal wins for all boards of the form $3 \times 6k$ for $k \in \mathbb{Z}_+$.

Table 6.1: Triomineering values for $3 \times n$ boards

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	V	V	F	F	F	H	F	V	F	F	F	H	F	F	F	F	F	H

6.1.2 Triomineering for $4 \times n$ boards

We find that the boards for $n = 6, 7, 8, 9, 10, 11$ have negative values. Further, $\{6, 7, 8, 9, 10, 11, 12\}$ forms a non-negative basis for $\mathbb{Z}_{\geq 6}$ by Theorem 5.2. Hence, Horizontal wins the game of Triomineering on $4 \times n$ boards for all $n \geq 6$ by Theorem 5.3. The following table summarizes the values for this game,

Table 6.2: Triomineering values for $4 \times n$ boards

n	1	2	3	4	5
	V	V	F	F	F

6.1.3 Triomineering for $5 \times n$ boards

In order for the game to be interesting, we must have that $n > 2$ as otherwise Horizontal would not be able to place any tiles. The following table summarizes the results obtained using the plug-in to CGS.

Table 6.3: Triomineering values for $5 \times n$ boards

n	1	2	3	4	5	6	7	8	9	10	11
	V	V	F	F	F	H	H	H	H	H	H

From the results obtained in Table 6.1.3, Theorem 5.2 tells us that $\{6, \ldots, 12\}$ is a non-negative basis for $\mathbb{Z}_{\geq 12}$. Hence, by Theorem 5.3, Horizontal wins all $5 \times n$ boards with $n \geq 12$.

6.2 Results on Tridomineering

6.2.1 Tridomineering for $2 \times n$ boards

Using CGS, we find that 4, 9, 11 and 14 have negative values. Furthermore, 4, 9, 11 and 14 form a non-negative generator set for $\mathbb{Z}_{\geq 12}$. The following table summarizes the values for boards whose width is less than 12. Therefore, by Theorem 5.3, we know who wins Tridomineering on boards of the form $2 \times n$ for arbitrary n.

Table 6.4: Tridomineering values for $2 \times n$ boards

n	1	2	3	4	5	6	7	8	9	10	11
	V	F	F	H	F	F	F	H	H	F	H

6.2.2 Tridomineering for $3 \times n$ boards

Using CGS, we find that 4, 5, and 6 have negative values. Furthermore, 4, 5, and 9 form a non-negative generator set for $\mathbb{Z}_{\geq 8}$. The following table summarizes the values for boards whose width is less than 8. So the winner of a game of Tridomineering on boards of the form $3 \times n$ is known for all n.

Table 6.5: Tridomineering values for $3 \times n$ boards

n	1	2	3	4	5	6	7
	V	F	F	H	H	H	F

6.2.3 Tridomineering for $4 \times n$ boards

The following table summarizes the results we have obtained so far. Horizontal wins on all boards of the form $4 \times 6k,\ k \in \mathbb{N}$.

Table 6.6: Tridomineering values for $4 \times n$ boards

n	1	2	3	4	5	6	7
	V	V	V	F	F	H	F

6.2.4 L-Tridomineering for $2 \times n$ boards

We present the winner of the game of L-Tridomineering on boards of the form $2 \times n$ for $n \in \{1, 2, \ldots, 13\}$.

Due to the above table, we were led to believe that the first player has a winning strategy for all $n > 13$. We tried to construct such an strategy by keeping symmetry,

Table 6.7: L-Tridomineering values for $2 \times n$ boards

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	V	F	F	F	F	F	F	F	F	F	F	F	F	F

that is, the first move was something close to dividing the board into two equal pieces whenever it was possible, or something close to it. However, such a technique does not work for $n = 30$. We need more experimentation to produce new conjectures on this matter.

6.3 Results on L-Tridomineering

6.3.1 L-Tridomineering for $3 \times n$ boards

Using CGS, we find that 4, 5, and 6 have negative values.

We show that there exists a winning strategy for all boards of the form $3 \times 4k$ for $k \in \mathbb{Z}_{>0}$. For that, we need the following lemma.

Lemma 6.1. *Let L be a subset –not necessarily proper– of a $m \times n$ board without holes (no squares removed in its interior), let G be the game of L-Tridomineering played on L, let L' be the board L with one square, or a domino placed vertically removed from the left or right boundary. Let G' be the game of L-Tridomineering played on L'. If $G < 0$ then $G' \not> 0$.*

Before looking at the proof of the lemma, note that a similar statement will hold by symmetry. Namely, if we know that $G > 0$ and we remove a square or domino from the last or first row of L, then $G' \not< 0$.

Proof. The statement is clear when we remove a domino placed vertically from anywhere in L, as $G < 0$, after any move from Vertical, Horizontal can always counter.

Now, suppose that L' is obtained from L by removing a square from the last or first column. Suppose $G < 0$ and $G' > 0$. Then, it is clear that Horizontal needs to cover the square that was removed to win; furthermore, he is able to use it since he wins G. One readily sees that the only moves that Horizontal can use to win on G are those that cannot be taken as moves on G'. For instance, Horizontal would not tile the square that was removed in L' with a straight triomino, as this would be the same as using a domino to tile L'. So the only moves that can help Horizontal win are dominoes and L-triominoes with the square that has been removed in the middle.

Now we proceed by induction. We assume that the lemma is true for all subboards of G. If Horizontal places a domino covering the square that has been removed, by induction, the resulting game G'' cannot be negative. If Horizontal is to win, then $G'' = 0$, which implies that Horizontal can win G' by placing a L-triomino, as showed in Figure 6.1-A, where the shadowed square represents the square that has been removed from L. This contradicts the assumption that $G' > 0$. It is clear that if Horizontal cannot place a L-triomino, then it is not true that $G' > 0$ and $G < 0$.

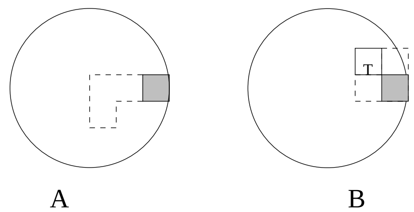

Figure 6.1: Proof of Lemma 6.1. The circle represents a general board. In this case a square from the right boundary has been removed.

The other option is that Horizontal places a L-triomino with the removed square in the middle, as depicted in Figure 6.1-B. If this is the case, the board resulting by removing the square formed by the L-triomino placed by Horizontal and T, as depicted in the figure, must be negative. This contradicts the fact that $G' > 0$. Again, if it is not possible to place an L-triomino, then it is not the case that $G' > 0$ and $G > 0$. ■

Although we are not certain that Theorem 5.3 holds for L-Tridomineering, the theorem is true for $3 \times n$ boards. The key here is the geometry of the board. Horizontal has the upper hand by tiling squares in the second row, and by doing this he reserves some squares in the first and third row for future moves.

Lemma 6.2. *Horizontal wins L-Tridomineering on $3 \times n$ boards where n is a linear combination of $4, 5$, and 6.*

Proof. As before, Horizontal will not tile using squares from both A and B'. If Vertical does the same, then it is clear that Horizontal will win.

If at some moment Vertical passes the border tiling squares from both A and B'; we proceed as follows: We know that if B is a board with a number of columns that is a linear combination of $4, 5$, and 6, then it can be divided into two pieces, one of them being a board A of the form $3 \times n$, where $n = 4, 5, 6$ and the other being a board B'. Without loss of generality, we can assume that A is contained in B' (this also includes the possibility that $A = B'$.) By induction, Horizontal wins on B'. We now show that Horizontal can win in $AB_{(1,1)}$. If Horizontal is to start the game, it would be better for him to start playing on A, as by placing a straight triomino in the second row, tiling the square neighboring with B' will certainly give him the upper hand. After that, Horizontal has guaranteed the victory on A and B', regardless of wether or not Vertical crosses the boundary. If he does cross the boundary, then the next move will be Horizontal's, and by the previous lemma, Horizontal can win on B'. Note that due to the properties of the $3 \times n$ board, using straight triominoes to tile the second row is of much advantage to Horizontal, because in that he can reserve at least two extra moves for each straight triomino that he places (the only board where Vertical has the same kind of advantage is the 3×2 board, which is a first player game. In fact, by covering the fourth column with a straight triomino, Vertical wins the 3×7 board if he plays first.)

Now, if Vertical starts the game, then Horizontal should counter by playing on the same board Vertical just did, unless he crossed a boundary. If Vertical did cross a boundary it would be better if Horizontal countered in B', assuming that B' properly contains A, since in this way Horizontal can guarantee at least two additional moves. If he cannot reserve at least two additional moves in B', then Horizontal should move on A after Vertical has crossed the boundary between A and B'. In each case, Horizontal is able to save enough extra moves to be the winner. ■

Table 6.8: L-Tridomineering values for $3 \times n$ boards

n	1	2	3	4	5	6	7
	V	F	F	H	H	H	F

6.4 Results on Lemineering

The game of Lemineering was also introduced in [BlaFra], and it consists of tiling a board with L-shaped triominoes, or L-triominoes for short (2×2 tiles with a corner removed). Table 6.9 presents the values of Lemineering for some small boards. Note that each number x on the table represents the nimber $*x$.

Table 6.9: Sprague-Grundy values for Lemineering on $m \times n$ boards.

m/n	1	2	3	4	5	6
1	0	0	0	0	0	0
2	0	1	2	0	3	1
3	0	2	0	1	2	2
4	0	0	1	0	1	0
5	0	3	2	1	0	1
6	0	1	2	0	1	1

Here we present two new results regarding this game. Denote by $G(m,n)$ where $m,n \in \mathbb{Z}_{>0}$ the game of Lemineering played on a $m \times n$ rectangular board and let $G'(m,n)$ be the game of Lemineering played on a $m \times (n+1)$ board with two opposite corner removed (which we assume to be the upper-left and lower-right corner.) With this notation in mind, we have the following result,

Theorem 6.3. *For non-negative integer n, $G(2,n)$ and $G'(2,n)$ have the same value.*

Proof.We need to prove that $G(2,n) + G'(2,n) = 0$. Since the values of each game equals its negative, that is equivalent to showing that $G(2,n) = G'(2,n)$.

Note that if the statement is true, then the second player can be guaranteed to be victorious. That is, no matter how the first player moves, the second player will always be able to perform a move as long as the first player is able to do so. We will go a little bit further and prove that the second player has a winning strategy on $G(2,n)+G'(2,n)$ so that

1. If the first player moves $(1,2;1)$ or $(1;1,2)$ on $G(2,n))$ then the second player counters on $G'(2,n)$ by playing $(1;1,2)$.

2. If the first player moves $(n-1, n; n)$ or $(n; n-1, n)$ on $G(2,n))$ then the second player counters on $G'(2,n)$ by playing $(n-1, n; n)$.
3. If the first player moves $(1; 1, 2)$ on $G'(2,n)$ then the second player counters on $G(2,n)$ by playing either $(1; 1, 2)$ or $(1, 2; 1)$.
4. If the first player moves $(n-1, n; n)$ on $G'(2,n)$ then the second player counters on $G(2,n)$ by playing either $(n; n-1, n)$ or $(n-1, n; n)$.

This rules are depicted in Figure 6.2.

We proceed by induction on n. For $n = 1, 2, 3$ the statement becomes trivial. Suppose that $G(2,k) + G'(2,k) = 0$ for all $3 < k \leq n$. We now show that the statement also holds for $k = n$.

If we label each row in increasing order with numbers from $[n]$, then we recognize the following cases,

(i) Each player only moves in the first $n-1$ rows. Then, by induction we have that

$$G(2,n) + G'(2,n) = G(2,n-1) + G'(2,n-1) = 0$$

(ii) At some point, the first player covers exactly one of the four squares labeled n. Suppose, without loss of generality, that he does so on $G(2,n)$. If this is the case, all the previous moves on $G(2,n)$ have been covered squares labeled at most $n-2$. Since by induction $G(2,n-2) + G'(2,n-2) = 0$ we know that the second player has been able to counter any move so far using squares labeled at most $n-2$. In other words, one can assume that the game played before tiling exactly one square from the last row of $G(2,n)$ has been just $G(2,n-2) + G'(2,n-2)$; that is, the second player has not had the need to use any square labeled n or $n-1$. Hence, the second player can counter by moving $(n-1; n-1, n)$. Once this move has been performed, the first player has no option that to play on squares labeled at most $n-2$, where we know the second player can always counter.

(iii) The first player covers two squares labeled n on the same board, say $G(2,n)$. This means that all the previous moves on $G(2,n)$ have tiled squares labeled at most $n-1$. Since $G(2,n)+G'(2,n-1)=0$, we also know by induction, that the second player has been able to counter any move so far tiling squares labeled at most $n-1$. So the second player can counter by moving $(n-1,n;n)$ on $G'(2,n)$. This move is still available, for otherwise one of the players moved $(n-2,n-1;n-1)$ on $G'(2,n)$. By induction, condition (4) guarantees that the following move was either $(n-2,n-1;n-1)$ or $(n-1;n-2,n-1)$, but if that is the case, then it is impossible for the first player to cover the two squares labeled n on $G(2,n)$. A similar analysis can be done to guarantee that if the first player covers both squares labeled n on $G'(2,n)$, then the second player can counter with a move that covers both of the two squares labeled n on $G(2,n)$. Any other move by the first player will have to be done on the squares labeled $n-1$ at most, where we know that the second player can counter.

This shows that the second player has a winning strategy for $G(2,n)+G'(2,n)$ satisfying $(1)-(4)$. In particular, one has that $G(2,n)+G'(2,n)=0$. ■

As was pointed out in [BlaFra], Lemineering is an impartial game and therefore every value that occurs in this game is a nimber. In other words, when playing Lemineering, the winner is either the first or second player. For instance, $G(2,4)$ is a second-player win. It is remarkable that by just removing a corner from a $2\times n$ board, Lemineering becomes a first-player game. To prove this, we use Theorem 6.3.

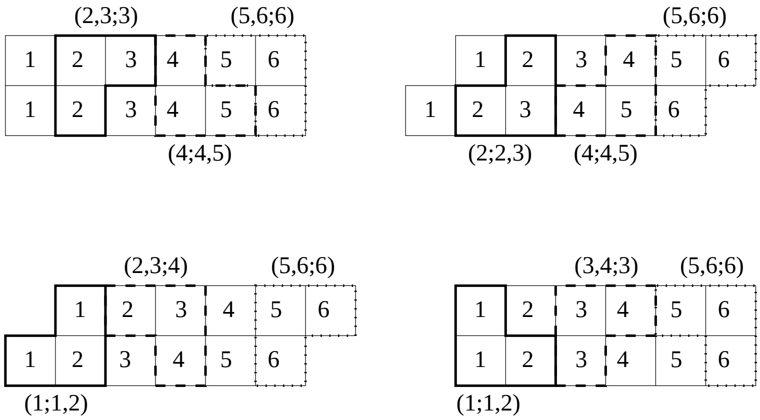

Figure 6.2: Examples of games $G(2,n) + G'(2,n)$. Note that the second player has a winning strategy satisfying $(1) - (4)$.

Theorem 6.4. *Let $B(n)$ be the value of the game of Lemineering played on a $2 \times n$ board with one corner removed. Then $B(n)||0$ for all n.*

Proof. Without loss of generality, we assume that the squares removed is the upper-right corner. After labeling both columns of the board with numbers from $[n]$ in increasing order, we now recognize two cases,

n **is odd** The first player moves $(\frac{n+1}{2} - 1, \frac{n+1}{2}; \frac{n+1}{2})$, this will bisect the board into two equal pieces. The value of Lemineering becomes the sum of the value of the game played on the two equal pieces. Since those values are the same nimber, they add up to 0, and so the first player (who is the second to play on $(\frac{n+1}{2} - 1, \frac{n+1}{2}; \frac{n+1}{2})$) can win this game.

n **is even** The first player moves $(\frac{n}{2}; \frac{n}{2}, \frac{n}{2} + 1)$. This will split the original board into two pieces: $G(2, \frac{n}{2} - 1)$ and $G'(2, \frac{n}{2} - 1)$, respectively. By Theorem 6.3, these two pieces have the same value when playing Lemineering, and so its sum is 0. So the first player (who is the second to play on $G(2, \frac{n}{2} - 1) + G'(2, \frac{n}{2} - 1)$) can win the game.

The last analysis proves that $B(n)||0$ for all n. ■

Chapter 7

Concluding Remarks

- For some of the boards, we need more experimental results. However, it is unlikely that mere computation will give the desired results. However, since we are dealing with a very particular kind of boards, perhaps some heuristic techniques can be used.
- We do not know if Theorem 5.3 can be applied in general for L-Tridomineering, although for the particular cases we have analyzed here, it seems plausible. One approach would be to generalize Lemma 6.1.

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