Cornell University Mathematics Department Senior Thesis

Continued Fractions

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Abstract

This paper examines some properties and theorems of continued fractions. The definitions, notations, and basic results are shown in the beginning. Then periodic continued fractions and best approximation are discussed in depth. Finally, a number of applications to mathematics, astronomy and music are examined.

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1 Definitions

A **continued fraction** is an expression in the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cfrac{b_5}{...}}}}} \tag{1}$$

In general, the numbers $a_1, a_2, a_3, ...$, $b_1, b_2, b_3, ...$ may be any real or complex numbers or functions of such variables. The number of terms can be finite or infinite. The discussion here will be restricted to **simple continued fractions** or **regular continued fractions**, which have the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{...}}}}} \tag{2}$$

where a_0 is an integer and the terms $a_1, a_2, ...$ are positive integers.

A **finite simple continued fraction** has only a finite number of terms, with the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{... + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}} \tag{3}$$

More precisely, this is an **n-th order continued fraction**. An n-th order continued fraction has $n + 1$ elements. The terms $a_0, a_1, a_2, a_3, ..., a_n$ are called the **partial quotients** of the continued fraction.

For simplicity, the simple continued fraction (2) can be called as **infinite continued fraction** while the finite simple continued fraction (3) can be called as **finite continued fraction**. For technical convenience, an infinite continued fraction (2) can be represented as $[a_0, a_1, a_2, ...]$, while a n-th order continued fraction (3) can be represented in the form $[a_0, a_1, a_2, ..., a_n]$.

A **segment** of the finite continued fraction $[a_0, a_1, a_2, ..., a_n]$ is

$$s_k = [a_0, a_1, a_2, ..., a_k] \tag{4}$$

where $0 \leq k \leq n$. Similarly, s_k can be a segment of the infinite continued fraction for arbitary $k \geq 0$.

A **remainder** of the finite continued fraction $[a_0, a_1, a_2, ..., a_n]$ is

$$r_k = [a_k, a_{k+1}, ..., a_n] \tag{5}$$

where $0 \leq k \leq n$. Similarly,

$$r_k = [a_k, a_{k+1}, a_{k+2}, ...] \tag{6}$$

can be a remainder of the infinite continued fraction for arbitary $k \geq 0$.

Every finite continued fraction $[a_0, a_1, a_2, ..., a_n]$ is the result of a finite number of rational operations on its elements, and can be represented as the ratio of two polynomials

$$\frac{P(a_0, a_1, a_2, ..., a_n)}{Q(a_0, a_1, a_2, ..., a_n)}$$

in $a_0, a_1, a_2, ..., a_n$ with integral coefficients. As long as $a_0, a_1, a_2, ..., a_n$ are numerical values, the given continued fraction can be represented as a normal fraction $\frac{p}{q}$. This representation is not unique, a way to solve this is as follows by induction. Notice that

$$[a_0, a_1, a_2, ..., a_n] = [a_0, r_1] = a_0 + \frac{1}{r_1} \tag{7}$$

Represent r_1 as

$$r_1 = \frac{p'}{q'}, \tag{8}$$

then

$$[a_0, a_1, a_2, ..., a_n] = a_0 + \frac{q'}{p'} = \frac{a_0 p' + q'}{p'} \tag{9}$$

r_1 is a finite segment that has already been calculated by induction. So it is possible to set

$$[a_0, a_1, a_2, ..., a_n] = \frac{p}{q} \tag{10}$$

$$r_1 = [a_1, a_2, ..., a_n] = \frac{p'}{q'},$$

As r_1 is also a continued fraction itself, this process can continue by the equations

$$p = a_0 p' + q' \tag{11}$$

$$q = p'$$

Denote by p_k/q_k the **canoncial representation** of the segment $s_k = [a_0, a_1, a_2, ..., a_k]$ and it is the **k-th order convergent** of the continued fraction. For an kth-order continued fraction γ, obviously $p_k/q_k = \gamma$.

2 The Representation of Rational Numbers by Continued Fractions

A *rational number* is a fraction of the form $\frac{p}{q}$ where p and q are integers with $q \neq 0$. Here is an example:

$$\frac{93}{35} = 2 + \frac{23}{35} = 2 + \cfrac{1}{\cfrac{35}{23}} = 2 + \cfrac{1}{1 + \cfrac{12}{23}} = 2 + \cfrac{1}{1 + \cfrac{1}{\cfrac{23}{12}}} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{11}{12}}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{12}{11}}}} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{11}}}} = [2, 1, 1, 1, 11] \quad (12)$$

It turns out that any rational number can be represented by a continued fraction, as stated by the following theorem.

Theorem 2.1 *Any finite simple continued fraction represents a rational number. Conversely, any rational number p/q can be represented as a finite simple continued fraction.*

Proof
The proof will be shown in Theorem 5.2.

Theorem 2.2 *Any rational number p/q can be represented as a finite simple continued fraction in which the last term can be modified so as to make the number of terms in the expansion either even or odd.*

Proof
Suppose that

$$\frac{p}{q} = [a_0, a_1, ..., a_n] \quad (13)$$

If $a_n = 1$, then

$$\cfrac{1}{a_{n-1} + \cfrac{1}{a_n}} = \frac{1}{a_{n-1} + 1} \quad (14)$$

and it is possible to represent

$$\frac{p}{q} = [a_0, a_1, ..., a_{n-2}, a_{n-1} + 1] \tag{15}$$

If $a_n > 1$, then

$$\frac{1}{a_n} = \frac{1}{(a_n - 1) + \frac{1}{1}} \tag{16}$$

and it is possible to represent

$$\frac{p}{q} = [a_0, a_1, ..., a_{n-2}, a_{n-1} - 1, 1] \tag{17}$$

QED

To avoid this ambiguity, the last partial quotient a_n of any finite continued fraction will be restricted to be greater than 1 from now on.

Now, consider the following:

$$\frac{35}{93} = 0 + \frac{35}{93} = 0 + \frac{1}{\frac{93}{35}} = 0 + \frac{1}{2 + \frac{23}{35}} = 0 + \frac{1}{2 + \frac{1}{\frac{35}{23}}} = 0 + \frac{1}{2 + \frac{1}{1 + \frac{12}{23}}}$$

$$= 0 + \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{23}{12}}}} = 0 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{11}{12}}}} = 0 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\frac{12}{11}}}}}$$

$$= 0 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{11}}}}} = [0, 2, 1, 1, 1, 11] \tag{18}$$

Compare this with $\frac{93}{35} = [2, 1, 1, 1, 11]$ in (12). In fact, it can be generalized to the following theorem:

Theorem 2.3 *For integers p, q where $p > q$, $\frac{p}{q} = [a_0, a_1, a_2, .., a_n]$ if and only if $\frac{q}{p} = [0, a_0, a_1, a_2, ..., a_n]$.*

Proof
If $p > q > 0$, then $\dfrac{p}{q} > 1$ and

$$\frac{p}{q} = [a_0, a_1, a_2, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{... + \cfrac{1}{a_n}}}}}} \tag{19}$$

where a_0 is an integer> 0. The reciprocal of $\dfrac{p}{q}$ is

$$\frac{q}{p} = \frac{1}{\frac{p}{q}} = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{... + \cfrac{1}{a_n}}}}}}} \tag{20}$$

$$= 0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{... + \cfrac{1}{a_n}}}}}}} = [0, a_0, a_1, a_2, ..., a_n]$$

Conversely, if $q < p$, then $\frac{q}{p}$ is of the form

$$\frac{q}{p} = 0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{... + \cfrac{1}{a_n}}}}}}} \tag{21}$$

and its reciprocal is

$$\frac{p}{q} = \cfrac{1}{\cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{... + \cfrac{1}{a_n}}}}}}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{... + \cfrac{1}{a_n}}}}}} \tag{22}$$

QED

This concludes the discussion of representing rational numbers by continued fractions.

3 Classical Theorems

The following theorem is a nice way to calculate p_k and q_k by recursion formulae, just using the previous two terms.

Theorem 3.1 *Law of formation of the convergents*
For any $k \geq 2$,

$$p_k = a_k p_{k-1} + p_{k-2} \tag{23}$$

$$q_k = a_k q_{k-1} + q_{k-2} \tag{24}$$

Proof
Proof by induction. When $k = 2$, it can be verified easily. Suppose they are valid for all $k < n$. Now consider the continued fraction $[a_1, a_2, ..., a_n]$ and denote its r^{th} convergent by $\frac{p'_r}{q'_r}$. By definition,

$$p_n = a_0 p'_{n-1} + q'_{n-1} \tag{25}$$

$$q_n = p'_{n-1} \tag{26}$$

and by inductive assumption

$$p'_{n-1} = a_n p'_{n-2} + p'_{n-3} \tag{27}$$

$$q'_{n-1} = a_n q'_{n-2} + q'_{n-3} \tag{28}$$

As a result,

$$p_n = a_0(a_n p'_{n-2} + p'_{n-3}) + (a_n q'_{n-2} + q_{n-3})$$

$$= a_n(a_0 p'_{n-2} + q'_{n-2}) + (a_0 p'_{n-3} + q_{n-3}) = a_n p_{n-1} + p_{n-2} \tag{29}$$

and

$$q_n = a_n p'_{n-2} + p'_{n-3} = a_n q_{n-1} + q_{n-2} \tag{30}$$

QED

It is convenient to define the convergent of order -1, set $p_{-1} = 1$ and $q_0 = 0$. With this convention, Theorem 3.1 is also true for $k = 1$.

Theorem 3.2 *For all $k \geq 0$,*

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k \tag{31}$$

Proof

Multiplying the first formula of Theorem 3.1 by q_{k-1} and the second one by p_{k-1}, then

$$p_k q_{k-1} = a_k p_{k-1} q_{k-1} + p_{k-2} q_{k-1} \tag{32}$$

$$q_k p_{k-1} = a_k q_{k-1} p_{k-1} + q_{k-2} p_{k-1} \tag{33}$$

Hence

$$q_k p_{k-1} - p_k q_{k-1} = -(p_{k-2} q_{k-1} - q_{k-2} p_{k-1}) \tag{34}$$

When $k = 0$,

$$q_0 p_{-1} - p_0 q_{-1} = 1(1) - a_0(0) = 1 \tag{35}$$

So the theorem is also true for $k = 0$ and the proof is complete.

QED

Corollary 3.3 *For all $k \geq 1$,*

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}} \tag{36}$$

Theorem 3.4 *For all $k \geq 1$,*

$$q_k p_{k-2} - p_k q_{k-2} = (-1)^{k-1} a_k \tag{37}$$

Proof

Multiplying the first formula of Theorem 3.1 by q_{k-2} and the second by p_{k-2}, the following equations are obtained.

$$p_k q_{k-2} = a_k p_{k-1} q_{k-2} + p_{k-2} q_{k-2} \tag{38}$$

$$q_k p_{k-2} = a_k q_{k-1} p_{k-2} + q_{k-2} p_{k-2} \tag{39}$$

Hence

$$q_k p_{k-2} - p_k q_{k-2} = -p_{k-2} q_{k-2} + q_{k-2} p_{k-2} = (-1)^{k-1} a_k \tag{40}$$

QED

Corollary 3.5 *For all $k \geq 2$,*

$$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = \frac{(-1)^{k-1} a_k}{q_k q_{k-2}} \tag{41}$$

Theorem 3.6 *The convergents of even order form an increasing sequence, while the convergents of odd order form a decresing sequence.*

Proof
This follows from Corollary 3.5.

QED

Theorem 3.7 *Every odd order convergent is greater in value than every even order convergent.*

Proof
This follows from Corollary 3.3.

QED

Theorem 3.8 *For arbitary k, $1 \leq k \leq n$,*

$$[a_0, a_1, ..., a_n] = \frac{p_{k-1} r_k + p_{k-2}}{q_{k-1} r_k + q_{k-2}} \tag{42}$$

Proof

$$[a_0, a_1, ..., a_n] = [a_0, a_1, ..., a_{k-1}, r_k] \tag{43}$$

The continued fraction on the right hand side of this equation has a $(k-1)$-st order convergent $\frac{p_{k-1}}{q_{k-1}}$. Its k-th order convergent, p_k/q_k, is the same as the fraction itself. Also, by Theorem 3.1,

$$p_k = r_k p_{k-1} + p_{k-2}$$

$$q_k = r_k q_{k-1} + q_{k-2}$$

Hence the proof is complete.

QED

Theorem 3.9 *For any $k \geq 1$,*

$$[a_0, a_1, a_2, ...] = \frac{p_{k-1}r_k + p_{k-2}}{q_{k-1}r_k + q_{k-2}} \tag{44}$$

Proof

The proof is the same as theorem 3.8.

QED

Theorem 3.10 *For any $k \geq 1$,*

$$[a_k, a_{k-1}, ...a_1] = \frac{q_k}{q_{k-1}} \tag{45}$$

Proof

For $k = 1$, it is trivial because

$$\frac{q_1}{q_0} = a_1$$

Suppose that $k > 1$ and

$$\frac{q_{k-1}}{q_{k-2}} = [a_{k-1}, a_{k-2}, ..., a_1].$$

By Theorem 3.1,

$$q_k = a_k q_{k-1} + q_{k-2}$$

So

$$\frac{q_k}{q_{k-1}} = a_k + \frac{q_{k-2}}{q_{k-1}} = \left[a_k, \frac{q_{k-1}}{q_{k-2}}\right].$$

Thus

$$\frac{q_k}{q_{k-1}} = [a_k, a_{k-1}, ...a_1]$$

and the theorem is proved.

QED

Theorem 3.11 *All convergents are irreducible.*

Proof

By Theorem 3.2, for all $k \geq 0$, $q_k p_{k-1} - p_k q_{k-1} = (-1)^k$. Any common divisor of the numbers p_k and q_k would at the same time be a divisor of the expression $q_k p_{k-1} - p_k q_{k-1}$, which is not possible.

QED

Theorem 3.12 *For any $k \geq 2$,*

$$q_k \geq 2^{\frac{k-1}{2}} \tag{46}$$

Proof

By the second formula of Theorem 3.1, $q_k > q_{k-1}$ for every $k \geq 2$. Hence the sequence

$$q_1, q_2, ..., q_n, ...$$

is always increasing. Then for all $k \geq 2$,

$$q_k = a_k q_{k-1} + q_{k-2} \geq q_{k-1} + q_{k-2} \geq 2q_{k-2}$$

Apply this inequality for k times, then

$$q_{2k} \geq 2^k q_0 = 2^k$$

for even number of terms and

$$q_{2k+1} \geq 2^k q_1 \geq 2^k$$

for odd number of terms.

QED

Thus the denominators of the convergents increase at least as rapidly as the terms of a geometric progression. This completes the discussion of classical results of continued fractions.

4 Periodic Continued Fractions

Definition 4.1 *A* ***purely periodic continued fraction*** *with* ***period*** *(or length) m is a continued fraction such that the initial block of partial quotients $a_0, ..., a_{m-1}$ is repeated over and over (so that $a_m = a_0, ..., a_{2m-1} = a_{m-1}$ and so on for each a_{km} where $k \geq 1$) and no shorter block $a_0, ..., a_{n-1}$ with $n < m$ has this property.*

A purely periodic continued fraction will be represented as $[\overline{a_0, a_1, ..., a_{m-1}}]$.

Definition 4.2 *A* ***periodic continued fraction*** *consists of an* ***initial block*** *of length n followed by a* ***repeating block*** *of length m. It is supposed that there is no shorter repeating block and the initial block does not end with a copy of the repeating block.*

A periodic continued fraction will be represented as $[a_0, a_1, ..., a_{n-1}, \overline{a_n, a_{n+1}, ..., a_{n+m-1}}]$.

Definition 4.3 *A* ***quadratic surd*** *is a solution of a quadratic equation $ax^2 + bx + c = 0$ with integer coefficients a, b, and c where $a \neq 0$ and its discriminant $b^2 - 4ac$ is not a perfect square.*

Theorem 4.4 *(Euler) If t is a periodic continued fraction, then t is a quadratic surd.*

Proof
Case 1: Assume t is purely periodic and has the form $[\overline{a_0, a_1, ..., a_{m-1}}]$. Then $t = r_0 = r_m = r_{2m} = ...$ and by Theorem 3.9,

$$[a_0, a_1, a_2, ...] = \frac{p_{m-1}r_m + p_{m-2}}{q_{m-1}r_m + q_{m-2}} = \frac{p_{m-1}t + p_{m-2}}{q_{m-1}t + q_{m-2}}$$

and so

$$q_{m-1}t^2 + (q_{m-2} - p_{m-1})t - p_{m-2} = 0$$

Since $m > 0$, $q_{m-1} \geq q_0 = 1$ and hence t is a quadratic surd.
Case 2: If t has initial terms and has the form $[a_0, a_1, ..., a_{n-1}, \overline{a_n, a_{n+1}, ..., a_{n+m-1}}]$, then $r_n = r_{n+m} = r_{n+2m} = ...$ and use Theorem 3.9 twice:

$$t = \frac{p_{n-1}r_n + p_{n-2}}{q_{n-1}r_n + q_{n-2}}$$

and

$$t = \frac{p_{n+m-1}r_n + p_{n+m-2}}{q_{n+m-1}r_n + q_{n+m-2}}$$

Then solve for r_n:
$$r_n = -\frac{q_{n-2}t - p_{n-2}}{q_{n-1}t - p_{n-1}}$$
and
$$r_n = -\frac{q_{n+m-2}t - p_{n+m-2}}{q_{n+m-1}t - p_{n+m-1}}$$
After some algebraic manipulation, it can be deduced that

$$(q_{n-2}q_{n+m-1} - q_{n-1}q_{n+m-2})t^2 + (-q_{n-2}p_{n+m-1} - p_{n-2}q_{n+m-1} + p_{n-1}q_{n+m-2} + q_{n-1}p_{n+m-2})t$$

$$+(p_{n-2}p_{n+m-1} - p_{n-1}p_{n+m-2}) = 0$$

If this is not a quadratic equation, then $q_{n-2}q_{n+m-1}$ will be equal to $q_{n-1}q_{n+m-2}$ and then q_{n+m-1} will divide $q_{n-1}q_{n+m-2}$. As q_{n+m-1} and q_{n+m-2} are relatively prime, q_{n+m-1} will have to divide q_{n-1} and this is not possible as $q_{n+m-1} > q_{n-1}$. Contradiction. Thus this is a quadratic equation and t is a quadratic surd.

QED

Theorem 4.5 *(Lagrange) If t is a quadratic surd, then t is a periodic continued fraction.*

Proof

Since t is a quadratic surd, there are integers $d_0 > 0$, b_0 and c_0 such that $d_0t^2 + b_0t + c_0 = 0$. By using Theorem 3.9 to replace t with an expression in an arbitary r_{k+1},

$$d_0(p_kr_{k+1} + p_{k-1})^2 + b_0(p_kr_{k+1} + p_{k-1})(q_kr_{k+1} + q_{k-1}) + c_0(q_kr_{k+1} + q_{k-1})^2 = 0$$

After some algebraic manipulations, this becomes

$$d_{k+1}r_{k+1}^2 + b_{k+1}r_{k+1} + c_{k+1} = 0$$

where
$$d_{k+1} = d_0p_k^2 + b_0p_kq_k + c_0q_k^2$$
$$b_{k+1} = 2d_0p_kp_{k-1} + b_0(p_kq_{k-1} + p_{k-1}q_k) + 2c_0q_kq_{k-1}$$
$$c_{k+1} = d_0p_{k-1}^2 + b_0p_{k-1}q_{k-1} + c_0q_{k-1}^2$$

such that $d_k = c_{k+1}$. If $d_{k+1} = 0$, this would not be a quadratic equation and r_{k+1} would be a rational number and then so would t, hence $d_{k+1} \neq 0$ for all $k \geq 0$. The discriminant is

$$b_{k+1}^2 - 4d_{k+1}c_{k+1} = (b_0^2 - 4d_0c_0)(p_kq_{k-1} - p_{k-1}q_k)^2$$

and thus remains unchanged.
As $|q_kt - p_k| < 1/q_{k+1} < 1/q_k$, it can be written that $p_k = q_kt + \epsilon/q_k$ where $|\epsilon| < 1$. Then

$$d_{k+1} = d_0(q_kt + \frac{\epsilon}{q_k})^2 + b_0(q_kt + \frac{\epsilon}{q_k})q_k + c_0q_k^2$$

$$= (d_0t^2 + b_0t + c_0)p_k^2 + \epsilon(2d_0t + b_0) + d_0(\frac{\epsilon}{q_k})^2$$

$$= \epsilon(2d_0t + b_0) + d_0(\frac{\epsilon}{q_k})^2$$

and note that

$$|d_{k+1}| < |2d_0t| + |d_0| + |b_0|$$

which means $|d_{k+1}|$ is bounded. As $d_k = c_{k+1}$, $|c_{k+1}|$ is also bounded and same for b_{k+1}, since the discriminant is a constant. Thus the coefficients of all the equations are bounded and there can be only finitely many such equations. Therefore, the equations will have to repeat and the corresponding r_{k+1}'s will coincide. Thus t is a periodic continued fraction and the proof is complete.

QED

Now the period length of a periodic continued fraction will be investigated. By quadratic formula, either solution to $ax^2 + bx + c = 0$ can be written as $t = \dfrac{(P_0 + \sqrt{D})}{Q_0}$ where $Q_0 \neq 0$, P_0 and $D > 0$ are integers such that D is not a perfect square and $Q_0|(D - P_0^2)$. To develop t as a continued fraction, set $a_0 = [t]$ and then

$$r_1 = \frac{1}{t - a_0} = \frac{Q_0}{P_0 - a_0Q_0 + \sqrt{D}} = \frac{P_1 + \sqrt{D}}{Q_1},$$

where

$$P_1 = a_0Q_0 - P_0$$

and

$$Q_1 = \frac{D - P_0^2 + 2a_0P_0Q_0 - (a_0Q_0)^2}{Q_0}$$

are integers. As $(D - P_1^2)/Q_1 = Q_0$, P_1 and Q_1 have the same divisibility property for P_0 and Q_0. Continue this process, it can be obtained that

$$r_{k+1} = \frac{P_{k+1} + \sqrt{D}}{Q_{k+1}}$$

where $P_{k+1} = a_kQ_k - P_k$ and $Q_{k+1} = (D - P_{k+1}^2)/Q_k$ is not zero since D is not a perfect square.
So $D - P_{k+1}^2 = Q_{k+1}Q_k$ and for sufficiently large k's such that

$$0 < P_{k+1} < \sqrt{D}$$

and

$$0 < \sqrt{D} - P_{k+1} < Q_{k+1} < \sqrt{D} + P_{k+1} < 2\sqrt{D}$$

As the inequalities $0 < P_{k+1} < \sqrt{D}$ and $0 < Q_{k+1} < 2\sqrt{D}$ make sure that there can be no more than $\sqrt{D} \times 2\sqrt{D}$ distinct pairs $\{P_{k+1}, Q_{k+1}\}$, the period length L of any surd of discriminant D satisfies $L(D) < 2D$. This is the Lagrange's estimate. The following theorem is a more precise statement.

Theorem 4.6 *Let* $t = \dfrac{(P_0 + \sqrt{D})}{Q_0}$ *be any quadratic surd where* $Q_0 \neq 0$, P_0 *and* $D > 0$ *are integers such that* D *is not a perfect square and* $Q_0|(D - P_0^2)$*. Then the length* L *of the repeating block in the periodic continued fraction of* t *satisfies*

$$L(D) = O(\sqrt{D} log(D)), \tag{47}$$

where O() is the Big Oh asymptotic notation.

Proof
From above, it suffices to estimate the number of pairs $\{P, Q\}$ of integers satisfying

$$0 < P < \sqrt{D}$$

$$0 < \sqrt{D} - P < Q < \sqrt{D} + P < 2\sqrt{D}$$

and

$$P^2 \equiv D \, mod \, Q$$

If $Q > \sqrt{D}$, then $\sqrt{D} < Q < \sqrt{D} + P < 2\sqrt{D}$ and hence

$$0 < Q - \sqrt{D} < P < \sqrt{D},$$

while if $Q < \sqrt{D}$, $0 < \sqrt{D} - P < Q < \sqrt{D}$ gives

$$0 < \sqrt{D} - Q < P < \sqrt{D}.$$

In either case, given a value of Q, the possible values for P are contained in an interval of less than Q, and the number of possible P's can be counted by their residue classes modulo Q. Thus

$$L(D) < \sum_{0<Q<2\sqrt{D}} \left(\sum_{P^2 \equiv D \; mod \; Q} 1 \right).$$

Consider the inner sum

$$\sum_{P^2 \equiv D \; mod \; Q} 1$$

for the 3 possible cases $(Q, D) = 1, 1 < (Q, D) < Q$ and $(Q, D) = Q$. In the first case, assume $P^2 \equiv D \, mod \, Q$ is solvable, that is, $D$is a *quadratic residue* of Q, and let the prime factorization of Q be

$$Q = 2^{k_0} p_1^{k_1} ... p_m^{k_m}$$

where $p_1, ..., p_m$ are distinct odd primes with positive integer exponents $k_1, ..., k_m$ and the integer exponent of 2 is $k_0 \geq 0$. Then any solution of $P^2 \equiv D \, mod \, Q$ must also satisfy the congruences

$$P^2 \equiv D \, mod \, 2^{k_0}, P^2 \equiv D \, mod \, p_1^{k_1}, ..., P^2 \equiv D \, mod \, p_m^{k_m}$$

If the number of solutions of N_i to each of these congruences $(i = 0, 1, 2, ..., m)$ is known, by the Chinese Remainder Theorem [9, P.194-207], there are exactly $N_0 N_1 ... N_m$ solutions to the original congruence.

For any of the odd primes p_i, a solution of $P^2 \equiv D \, mod \, p_i^{k_i}$ means there is a solution of $P^2 \equiv D \, mod \, p_i$, and there are exactly two such solutions. By Euler's lemma, if there exists a number k such that $k^2 \equiv a (mod \, p)$, then

$a^{(p-1)/2} \equiv 1(mod\, p)$. In Legendre symbol, if p is an odd prime, then for every c,

$$\left(\frac{c}{p}\right) \equiv c^{(p-1)/2}(mod\, p).$$

For a more complete discussion, see [9, P.401].
Thus the solutions pull back to two solutions modulo $p_i^{k_i}$ and so $N_i = 2$ for $i = 1, ..., m$. For the prime 2, the situation is slightly different. Any odd number a is always a square modulo 2, and a is square mdulo 4 only if $a \equiv 1(mod 4)$, and a is a square modulo 2^z only if $a \equiv 1(mod\, 8)$ for $z \geq 3$. Thus given a solution of $P^2 \equiv D\, mod\, 2$, there will be two solutions modulo 4 and four solutions modulo 2^z for $z \geq 3$. For the case $(Q, D) = 1$,

$$\sum_{P^2 \equiv D\ mod\ Q} 1 = 2^k 2^m,$$

where $k = 0$ if $k_0 \leq 1$, $k = 1$ if $k_0 = 2$ and $k = 2$ if $k_0 \geq 3$, and m is the number of distinct odd prime divisors of Q. Since $2^{k+m} \leq \tau(Q)$, where $\tau(n)$ is the number of divisors of the integer n, it is shown that

$$\sum_{P^2 \equiv D\ mod\ Q} 1 \leq \tau(Q)$$

The contribution to this sum for a Q in the second case, when $1 < (Q, D) < Q$, can be no more than that from the first case since for these Q's the congruence $P^2 \equiv D\, mod\, Q$, is the same as

$$(Q, D)\left(\frac{P}{(Q, D)}\right)^2 \equiv \frac{D}{(Q, D)}\left(mod\ \frac{Q}{(Q, D)}\right)$$

which either has no solutions or at most $\tau(Q/(Q, D))$ solutions, as in the first case. Finally if Q divides D,

$$\sum_{P^2 \equiv D\ mod\ Q} 1 = O(\sqrt{Q}).$$

Since

$$\sum_{k=1}^{n} \tau(k) = \sum_{d=1}^{n} \left[\frac{n}{d}\right] = n log(n) + O(n),$$

the estimate that

$$L = O(\sum_{0<Q<2\sqrt{D}} \tau(Q))$$

is equivalent to

$$L(D) = O(\sqrt{D} log(D)).$$

Hence the theorem is established.

QED

5 Best Approximation

Theorem 5.1 *There is a unique continued fraction for every real number α.*

Proof

It will be proved by mathematical induction. Suppose α has two continued fraction representations

$$\alpha = [a_0, a_1, a_2, ...] = [a_0', a_1', a_2', ...], \tag{48}$$

where they can be infinite or finite. It is trivial that $a_0 = a_0' = \lfloor \alpha \rfloor$, where $\lfloor \alpha \rfloor$ is the greatest integer $\leq \alpha$. Assume that

$$a_i = a_i' \tag{49}$$

for $0 \leq i \leq n$, then in analogous notation,

$$p_i = p_i' \tag{50}$$

and

$$q_i = q_i' \tag{51}$$

for $0 \leq i \leq n$. By Theorem 3.9,

$$\alpha = \frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}} = \frac{p_n' r_{n+1}' + p_{n-1}'}{q_n' r_{n+1}' + q_{n-1}'} = \frac{p_n r_{n+1}' + p_{n-1}}{q_n r_{n+1}' + q_{n-1}} \tag{52}$$

and hence $r_{n+1} = r_{n+1}'$. As $a_{n+1} = \lfloor r_{n+1} \rfloor$ and $a_{n+1}' = \lfloor r_{n+1}' \rfloor$, $a_{n+1} = a_{n+1}'$. By induction, all partial quotients of the two continued fractions are the same. Hence the representation of real numbers by continued fractions is unique, with the exception in Theorem 2.2, where $a_{n+1} = 1, r_n = a_n + 1$ and $a_n \neq \lfloor r_n \rfloor$. This is excluded already in Section 2.

QED

With this theorem, many important results can be derived. The following theorem is the generalization of Theorem 2.1.

Theorem 5.2 *The continued fraction corresponds to a real number α is finite if α is rational, it is infinite if α is irrational.*

Proof

Denote the largest integer not exceeding α by a_0. If α is not an integer, then

$$\alpha = a_0 + \frac{1}{r_1}, \tag{53}$$

where $r_1 > 1$ as

$$\frac{1}{r_1} = \alpha - a_0 < 1 \tag{54}$$

The process continues as long as r_n is not an integer. Denote the greatest integer not larger than r_n by a_n, and define r_{n+1} by

$$r_n = a_n + \frac{1}{r_{n+1}} \tag{55}$$

Note that $r_n > 1$. By (7),

$$\alpha = [a_0, r_1] \tag{56}$$

Assume that

$$\alpha = [a_0, a_1, ..., a_{n-1}, r_n], \tag{57}$$

then

$$\alpha = [a_0, a_1, ..., a_{n-1}, a_n, r_{n+1}] \tag{58}$$

So (57) is valid for all n, assuming $r_1, ..., r_{n-1}$ are not integers.
If α is a rational number, then all r_i's will be rational as they are obtained by subtraction and division. It will be shown that the process will stop in a finite number of steps. Suppose $r_n = a/b$. Then

$$r_n - a_n = \frac{a - ba_n}{b} = \frac{w}{b}, \tag{59}$$

where $w < b$ as $r_n - a_n < 1$. By (55),

$$r_{n+1} = \frac{b}{w} \tag{60}$$

where $w \neq 0$, or else r_n is an integer and the process stops. Hence r_{n+1} has a smaller denominator than r_n. Consider $r_1, r_2, r_3, ...$, eventually there must be some i such that $r_i = a_i$. Then (57) shows that α is represented by a finite continued fraction with the last partial quotient $a_n = r_n > 1$.

If α is an irrational number, then all r_i's are irrational and the process never comes to an end. Consider

$$\frac{p_n}{q_n} = [a_0, a_1, a_2, ..., a_n], \tag{61}$$

where $q_n > 0$ and p_n, q_n have no common factors. By (57) and Theorem 3.9,

$$\alpha = \frac{p_{n-1}r_n + p_{n-2}}{q_{n-1}r_n + q_{n-2}} \tag{62}$$

for $n \geq 2$, and

$$\frac{p_n}{q_n} = \frac{p_{n-1}a_n + p_{n-2}}{q_{n-1}a_n + q_{n-2}}. \tag{63}$$

Then

$$\alpha - \frac{p_n}{q_n} = \frac{(p_{n-1}q_{n-2} - q_{n-1}p_{n-2})(r_n - a_n)}{(q_{n-1}r_n + q_{n-2})(q_{n-1}a_n + q_{n-2})} \tag{64}$$

and hence

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{(q_{n-1}r_n + q_{n-2})(q_{n-1}a_n + q_{n-2})} < \frac{1}{q_n^2}. \tag{65}$$

So $\dfrac{p_n}{q_n}$ tends to α when n tends to infinty. This means the infinite continued fraction $[a_0, a_1, a_2, ...]$ is equal to α.

QED

Definition 5.3 *A rational number a/b is a **best approximation** of the number t if $|bt - a| < |qt - p|$ for any rational number $p/q \neq a/b$ with $0 < q \leq b$.*

Theorem 5.4 *Every best approximation to a real number t is a convergent of the continued fraction of t.*

Proof

Let $t = [a_0, a_1, a_2, ...]$. If t is rational, assume the final partial quotient $a_n \geq 2$. Suppose a/b is not a convergent, or else it is done already. Suppose $a/b < p_0/q_0$.

Then

$$|bt - a| = b\left|t - \frac{a}{b}\right| \geq \left|t - \frac{a}{b}\right| > |t - a_0| \tag{66}$$

which contradicts that a/b is a best approximation. Then assume that $a/b > p_1/q_1$. Since $p_1/q_1 > t$,

$$|bt - a| = b\left|t - \frac{a}{b}\right| > b\left|\frac{a_1}{b_1} - \frac{a}{b}\right| \geq b\frac{1}{q_1 b} = \frac{1}{q_1} = \frac{1}{a_1} \tag{67}$$

But this is also a contradiction since $|t - a_0| < 1/a_1$. So if a/b lies between p_0/q_0 and p_1/q_1 and it is not a convergent, it lies between two convergents of the same order, say k and $k+2$. Then

$$\frac{1}{q_k q_{k+1}} = \left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| > \left|\frac{a}{b} - \frac{p_k}{q_k}\right| \geq \frac{1}{q_k b} \tag{68}$$

and so $b > q_{k+1}$. As

$$|bt - a| = b\left|t - \frac{a}{b}\right| > b\left|\frac{a_{k+2}}{b_{k+2}} - \frac{a}{b}\right| \geq b\frac{1}{q_{k+2} b} = \frac{1}{q_{k+2}}, \tag{69}$$

the lower bounds on the size of b and the degree of approximation to t are both a/b. By Theorem 3.1, $|q_{k+1}t - p_{k+1}| < 1/q_{k+2}$, so a/b cannot be a best approximation. The proof of the theorem is complete.

QED

Theorem 5.5 *If a real number t is not of the form $a_0 + \frac{1}{2}$, then every convergent of t is a best approximation to t.*

Proof

Let q_n be given and Q be the value of b such that $0 < b \leq q_n$ and $|bt - a|$ is minimized (take the smallest one if there are several choices for Q). Let P be the corresponding value for a and suppose it is unique. P/Q is a best approximation and hence must be a convergent p_k/q_k for some $k \leq n$ by Theorem 5.4. If $k < n$, by Theorem 3.1,

$$|q_k t - p_k| > \frac{1}{q_k + q_{k+1}} \geq \frac{1}{q_k + q_{k-1}}, \tag{70}$$

and $|q_n t - p_n| < 1/q_{n+1}$.

If $|q_k t - p_k| < |q_n t - p_n|$, then $q_{n+1} < q_n + q_{n-1}$, which is impossible. Hence $k = n$ and P/Q is the convergent p_n/q_n.

If P is not unique, then Qt is between two integers, say P and $P+1$. Then $Qt-P = 1/2 \Rightarrow t = (1+2P)/(2Q)$. As $(1+2P, 2Q) \neq 2$, if $(1+2P, 2Q) = m > 1$, then $(2Q/m)t - (1+2P)/m = 0$, and this contradicts the definition of Q. So the continued fraction of t has ending with $t = p_n/q_n$, where $p_n = 1+2P$ and $q_n = 2Q = a_n q_{n-1} + q_{n-2}$ with $a_n \geq 2$. If $n = 1$ and $a_n > 2$ or $n > 1$, then $q_{n-1} < Q$ so that

$$|q_{n-1}t - p_{n-1}| = \left| q_{n-1}\frac{p_n}{q_n} - p_{n-1} \right| = \frac{1}{q_n} = \frac{1}{2Q} \leq |Qt - P| = \frac{1}{2} \tag{71}$$

which contradicts the definition of Q. Since $n = 1$ and $a_n = 2$ are excluded in the theorem statement, the theorem is established.

QED

The following theorem follows directly from the previous two theorems. A proof is provided here independently.

Theorem 5.6 *Each convergent is nearer to the value of an infinite continued fraction than is the preceding convergent.*

Proof

Let the continued fraction of the given irrational number x be

$$x = [a_0, a_1, ..., a_n, x_{n+1}]. \tag{72}$$

where

$$x_{n+1} = [a_{n+1}, a_{n+2}, ...]. \tag{73}$$

By Theorem 3.9,

$$x = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}}, \tag{74}$$

hence

$$x_{n+1}(xq_n - p_n) = -q_{n-1}x + p_{n-1} = -q_{n-1}\left(x - \frac{p_{n-1}}{q_{n-1}}\right). \tag{75}$$

for $n \geq 1$. Divide both sides by $x_{n+1}q_n$:

$$x - \frac{p_n}{q_n} = \left(-\frac{q_{n-1}}{x_{n+1}q_n}\right)\left(x - \frac{p_{n-1}}{q_{n-1}}\right) \tag{76}$$

Take absolute values on both sides:

$$\left|x - \frac{p_n}{q_n}\right| = \left|\frac{q_{n-1}}{x_{n+1}q_n}\right| \left|x - \frac{p_{n-1}}{q_{n-1}}\right| \tag{77}$$

For $n \geq 1$, $x_{n+1} > 1$, and $q_n > q_{n-1} > 0$; so

$$0 < \frac{q_{n-1}}{x_{n+1}q_n} < 1 \Rightarrow 0 < \left|\frac{q_{n-1}}{x_{n+1}q_n}\right| < 1 \tag{78}$$

Hence

$$\left|x - \frac{p_n}{q_n}\right| < \left|x - \frac{p_{n-1}}{q_{n-1}}\right| \tag{79}$$

and the theorem is established.

QED

This important theorem makes the applications in the next section possible.

6 Applications

In this section, a few applications will be examined, based on the theories discussed in the previous sections.

6.1 Quadratic Equations

There are many ways to solve quadratic equations, like using quadratic formula, completing the square, normal factoring and so on. One rather special method involves continued fraction. This method requires more calculations and can only obtain one of the two roots for most of the time. Hence it is less commonly used. Consider $x^2-6x+8=0$, which has the roots 2 and 4. The process begins by writing

$$x = 6 - \frac{8}{x} \tag{80}$$

Substitute (80) into x on the right hand side:

$$x = 6 - \cfrac{8}{6 - \cfrac{8}{x}} \tag{81}$$

Do this again:

$$x = 6 - \cfrac{8}{6 - \cfrac{8}{6 - \cfrac{8}{x}}} \tag{82}$$

This process can generate an infinite continued fraction. One of the roots can be approximated. Set $x = 3$ in the right hand side of (80), then $x = 10/3 = 3.\dot{3}$. Put $x = 10/3$ in the right hand side of (81), then $x = 34/9 = 3.\dot{7}$. Put $x = 34/9$ in the right hand side of (82), then $x = 258/65 \approx 3.96923$. The answer gets closer and closer to 4, which is one of the roots. By the first program in Appendix A, the tables on the next page can be obtained.
However, the other root 2 cannot be calculated with this method. This can be generalized to solve any quadratic equation.
The equation $34x^2+3254x-549 = 0$ has roots 0.1684190506 and -95.8743014036. It is solved below in just 4 steps.

Table 1: Solving $x^2 - 6x + 8 = 0$, using 3 as initial value

step	result
1	3.33333333333333333
2	3.77777777777777778
3	3.96923076923076923
4	3.99804878048780488
5	3.99993896670633831
6	3.99999904632613834
7	3.99999999254941943
8	3.99999999997089617
9	3.99999999999994316
10	3.99999999999999994

Table 2: Solving $x^2 - 6x + 8 = 0$, using 17 as initial value

step	result
1	5.52941176470588235
2	4.24299065420560748
3	4.02745512143611404
4	4.00169414217762429
5	4.00005289853450714
6	4.00000082651808245
7	4.00000000645716987

Table 3: Solving $34x^2 + 3254x - 549 = 0$, using 9452.183 as initial value

step	result
1	-95.7041740640915499
2	-95.8743008776620645
3	-95.8743014035849802
4	-95.8743014035849774

6.2 Calendar Construction

The construction of a calendar that accurately determines the seasons by counting the days is an important issue for human beings since ancient time. Seasons depend on the revolution on the Earth around the Sun while days depend on the rotation of the Earth about its axis.

The Julian calendar used the approximation $365\frac{1}{4}$ days for 1 year. It was carried out by extending one extra day every four "common" years to form a "leap" year. After 1600 years, the error accumulated to 10 days. Pope Gregory XIII revised it by omitting one leap year every century except every fourth century. This is based on the approximation $365\frac{97}{400}$ days for a year. The Gregorian calendar is more accurate yet simple to use. By definition, a tropical year (the time it takes for the Earth to revolve around the Sun once) is

$$\frac{315569259747}{864000000} = [365, 4, 7, 1, 3, 5, 6, 1, 1, 3, 1, 7, 7, 1, 1, 1, 1, 2] \tag{83}$$

days long and the construction of a calendar reduces to selecting an approximation of the error

$$c = \frac{7750361}{32000000} = [0, 4, 7, 1, 3, 5, 6, 1, 1, 3, 1, 7, 7, 1, 1, 1, 1, 2] \tag{84}$$

between the tropical year and the common year. The first few convergents of c are in the following table:

Table 4: The first few convergents of the error between tropical and common year

k	p_k/q_k
0	$\frac{0}{1}$
1	$\frac{1}{4}$
2	$\frac{7}{29}$
3	$\frac{8}{33}$
4	$\frac{31}{128}$
5	$\frac{163}{373}$

So the Julian calendar is just a realization of the first convergent. If our notation system was based on powers of 2 instead of powers of 10, it could be possible to design a calendar in which leap years occur every fourth year with every thirty-second leap year omitted. The annual error would be

$$c - \frac{31}{128} \approx 0.00001128 \tag{85}$$

which amounts to be the loss of one day every hundred thousand years. However, this is not as easy to use as the Julian calendar. In fact, nobody used this calendar. Actually it is not proposed anywhere in the world besides in Russia by Russian astronomer Medler in 1864.

6.3 Astronomy

Dutch physicist and astronomer Christiaan Huygens designed an *automatic planetarium* to show relative motions of the six known planets around the sun (and separately, the Moon around the Earth). Each model planet was attached to a large circular gear which was driven by a smaller gear mounted on a common drive shaft. Using the best data available in 1680 and approximating the year as $365\frac{35}{144}$ days, Huygens calculated the ratios of the periods of the planet's orbits to the Earth's year and obtained the following values.

Table 5: Gear ratios of the solar system used by Huygens

planet	period	convergent	gear teeth
Mercury ☿	$\frac{25335}{105190}$	$\frac{p_5}{q_5} = \frac{33}{137}$	33:137
Venus ♀	$\frac{64725}{105190}$	$\frac{p_5}{q_5} = \frac{8}{13}$	32:52
Earth ♁	1		
Mars ♂	$\frac{197836}{105190}$	$\frac{p_5}{q_5} = \frac{79}{42}$	158:84
Jupiter ♃	$\frac{1247057}{105190}$	$\frac{p_3}{q_3} = \frac{83}{7}$	166:137
Saturn ♄	$\frac{3095277}{105190}$	$\frac{p_1}{q_1} = \frac{59}{2}$	118:4

After expanding these ratios as continued fractions, he chose gear pairs corresponding to convergents such that the component numbers would be practical to construct as gear teeth.
After Huygens died, two more planets Uranus and Neptune are discovered. Their gear ratios are calculated by the second program in Appendix A.

Table 6: Gear ratios of Uranus and Neptune

planet	period	ratio	convergent	gear teeth
Uranus ♅	30707.4896	$\frac{30707489600}{365256366}$	$\frac{p_1}{q_1} = \frac{1177}{14}$	1177:14
Neptune ♆	60223.3528	$\frac{60223352800}{365256366}$	$\frac{p_2}{q_2} = \frac{1319}{8}$	1319:8

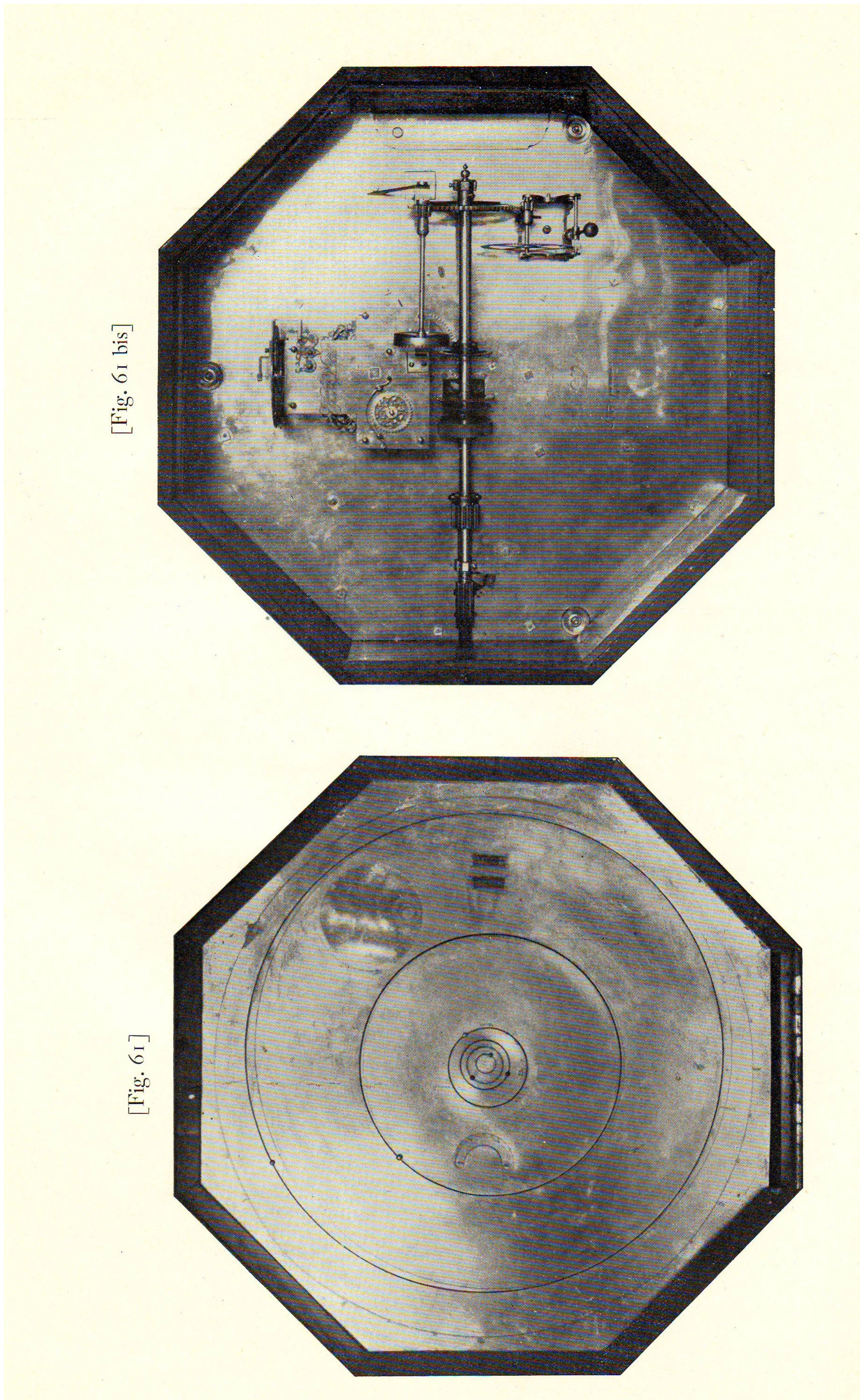

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Figure 1: Photograph of Huygen's automatic planetarium [27]

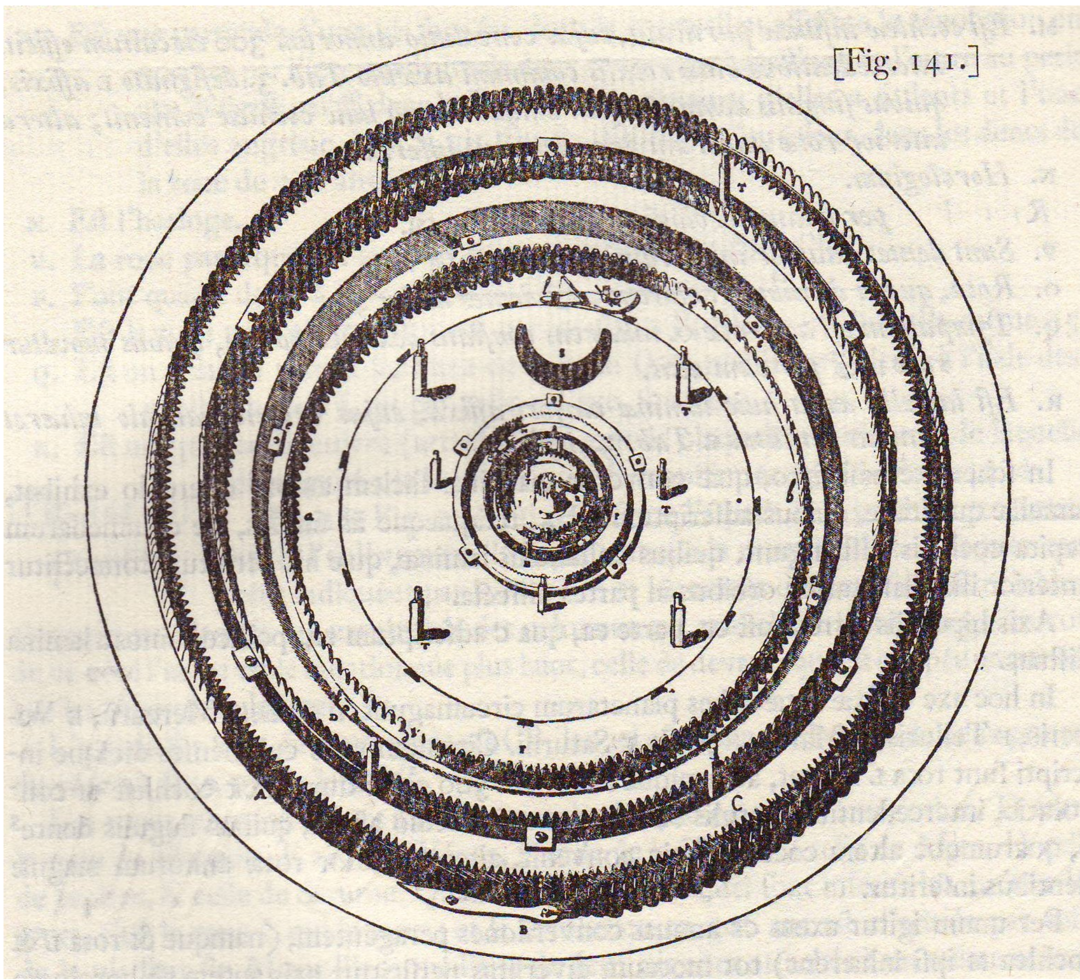

Figure 2: Sketch of Huygen's automatic planetarium (front) [27]

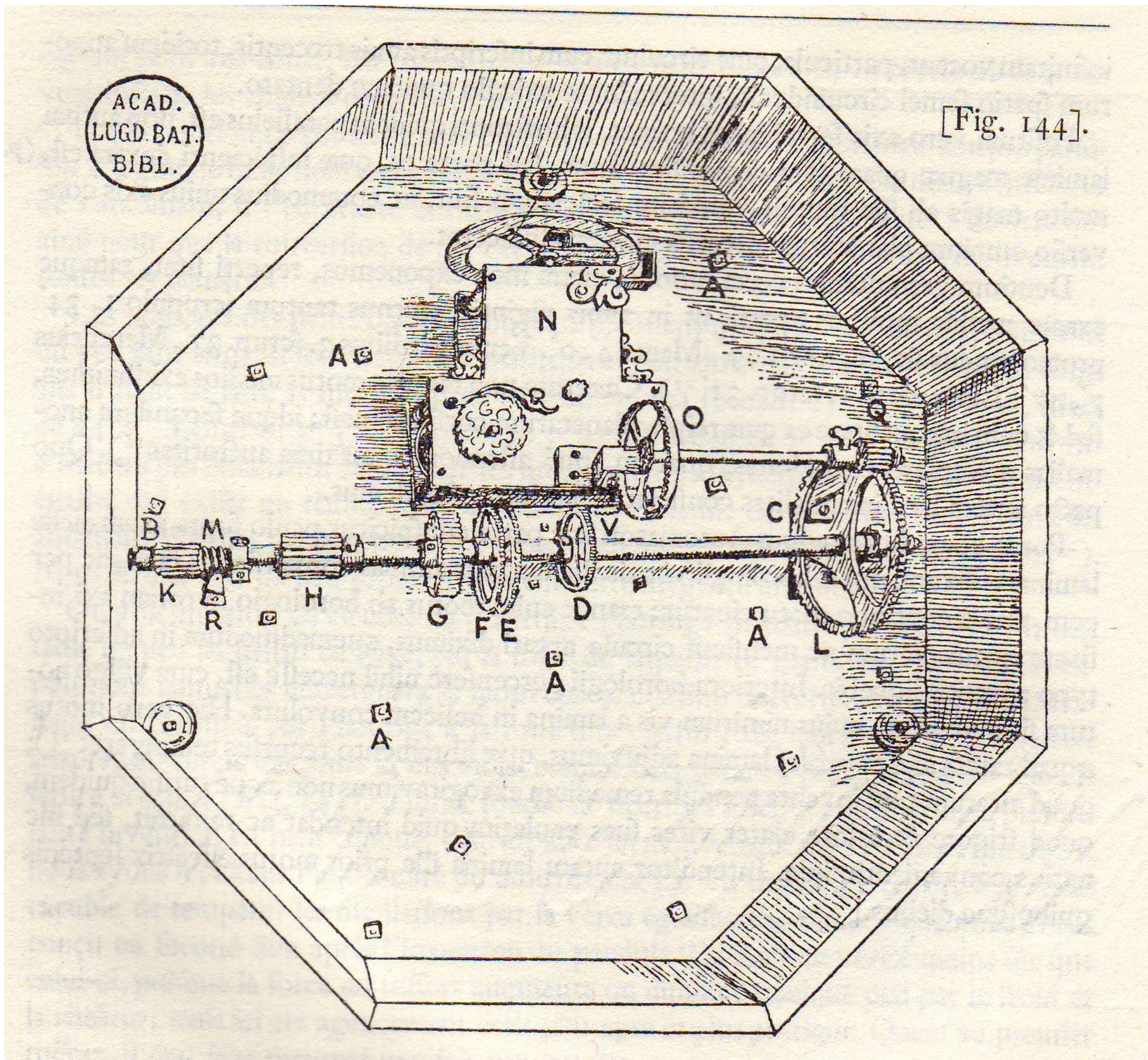

Figure 3: Sketch of Huygen's automatic planetarium (back) [27]

Figure 4: Eris, Pluto, Ceres, Moon and Earth [53]

Table 7: Gear ratios of dwarf planets

planet	period	ratio	convergent	gear teeth
Ceres	1679.819	$\frac{1679819000}{365256366}$	$\frac{p_3}{q_3} = \frac{23}{5}$	115:25
Pluto ♇	90613.3055	$\frac{90613305500}{365256366}$	$\frac{p_1}{q_1} = \frac{2977}{12}$	2977:12
Eris	203500	$\frac{20350000}{36525}$	$\frac{p_1}{q_1} = \frac{3343}{6}$	3343:6

There are also three dwarf planets in the Solar System. They are Ceres, Pluto, and Eris.
Note that as the periods of Pluto and Eris is very large comparing to Earth, it is not quite possible to provide reasonable gear ratios. The same holds for Uranus and Neptune.

In this paper, the periods of exoplanet systems of Gliese 876 and Mu Arae and satellites of Saturn and Uranus are also investigated.

Gliese 876 is a red dwarf star located approximately 15 light-years away in the constellation Aquarius. Currently, it is known to host three extrasolar planets.

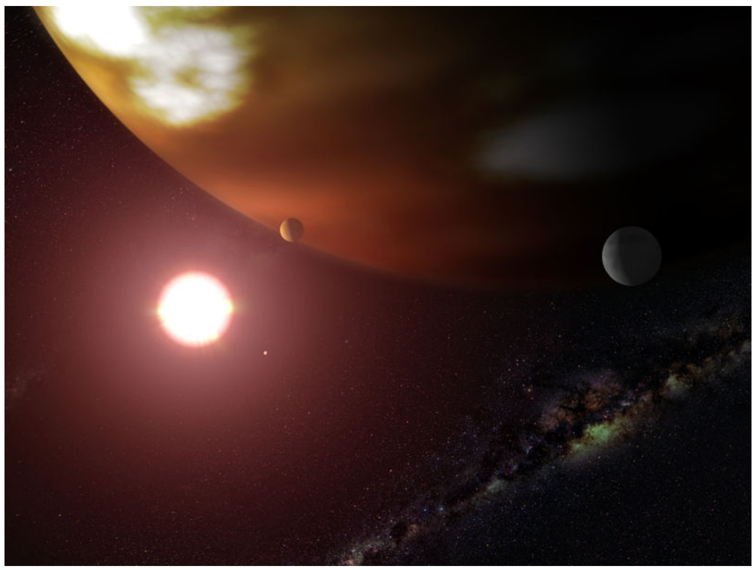

Figure 5: Gliese 876 planetary system [21]

The period of GJ 876 c is set to 1 in order for comparison. The gear teeth are calculated by the second program in Appendix A by using convergents.

Table 8: Gear ratios of the planetary system of Gliese 876

extrasolar planet	orbital period(days)	ratio	convergent	gear teeth
GJ 876 c	30.340	1	-	-
GJ 876 b	60.940	$\frac{6094}{3034}$	$\frac{p_1}{q_1} = \frac{233}{116}$	233:116
GJ 876 d	1.937760	$\frac{193776}{3034000}$	$\frac{p_4}{q_4} = \frac{3}{47}$	9:141

Mu Arae is a Sunlike yellow-orange star located around 50 light years away in the constellation Ara. The star has a planetary system with four known planets. The period of mu Ara e is set to 1.

Figure 6: Mu Arae [35]

Table 9: Gear ratios of the planetary system of Mu Arae

extrasolar planet	orbital period(days)	ratio	convergent	gear teeth
mu Ara e	310.55	1	-	-
mu Ara b	630.0	$\frac{63000}{31055}$	$\frac{p_2}{q_2} = \frac{71}{35}$	71:35
mu Ara c	2490	$\frac{249000}{31055}$	$\frac{p_1}{q_1} = \frac{441}{55}$	441:55
mu Ara d	9.550	$\frac{955}{31055}$	$\frac{p_3}{q_3} = \frac{2}{65}$	4:130

Similarly, the same method can be applied to the moons of Saturn (♄). Saturn has 60 moons as of now (57 of them are confirmed and named).

Figure 7: The Saturnian system (photographic montage) [46]

Here some of them are considered. The period of the largest moon, Titan, is set to 1 here for comparison.

Table 10: Gear ratios of the Saturnian system

	moon	orbital period(days)	ratio	convergent	gear teeth
VI	Titan	15.945421	1	-	-
I	Mimas	0.9424218	$\frac{9424218}{159454210}$	$\frac{p_3}{q_3} = \frac{12}{203}$	12:203
II	Enceladus	1.370218	$\frac{1370218}{15945421}$	$\frac{p_5}{q_5} = \frac{11}{128}$	11:128
VIII	Iapetus	79.330183	$\frac{79330183}{15945421}$	$\frac{p_2}{q_2} = \frac{199}{40}$	199:40
IX	Phoebe	550.564636	$\frac{550564636}{15945421}$	$\frac{p_2}{q_2} = \frac{69}{2}$	138:4

Table 11: Gear ratios of the Uranian system

	moon	orbital period(days)	ratio	convergent	gear teeth
I	Ariel	2.520379	1	-	-
II	Umbriel	4.144177	$\frac{4144177}{2520379}$	$\frac{p_6}{q_6} = \frac{171}{104}$	171:104
III	Titania	8.705872	$\frac{8705872}{2520379}$	$\frac{p_3}{q_3} = \frac{38}{11}$	156:44

Uranus (♅) is the seventh planet from the Sun, and the third largest planet in size. It has 27 known satellites. The period of Uranus I is set to 1 for comparison.

6.4 Music

In music, an octave is the interval between one musical note and another with double or half its frequency. The ancient Greeks realized that sounds which have frequencies in rational proportion are perceived as harmonious. The great scientist and philosopher Pythagoras noticed that subdividing a vibrating string into rational proportions produces consonant sounds. This is because the length of a string is inversely proportional to its fundamental frequency. If basic frequencies a and b have ratio $a/b = m/n$ for some small integers m and n, the sound will be consonant as they will have overtones in common. The ancient Greeks found the consonance of "octaves" ($a/b = 2/1$) and "perfect fifths" ($a/b = 3/2$). They combined these to get a scale, but an approximation is needed to keep the scale finite. They had to find a power of 3/2 to approximate a power of 2.

$$\left(\frac{3}{2}\right)^m = 2^n \Rightarrow \frac{3}{2} = 2^{n/m} \Rightarrow \frac{b}{a} = \frac{log(\frac{3}{2})}{log(2)} \tag{86}$$

Figure 8: Octave [36]

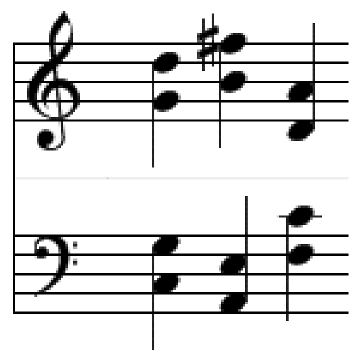

Figure 9: Perfect fifth [38]

It turns out that

$$\frac{log(\frac{3}{2})}{log(2)} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{23+\cfrac{1}{2+...}}}}}}}}}} \tag{87}$$

Table 12: The first few convergents of $log(\frac{3}{2})/log(2)$

k	p_k/q_k
1	$\frac{1}{2}$
2	$\frac{3}{5}$
3	$\frac{7}{12}$
4	$\frac{24}{41}$

It is the approximation $7/12 = 0.58\dot{3}$ which suggests an octave of 12 steps, with a perfect fifth equal to 7 semi-tones. If 5 is chosen then there will be too few notes, while choosing 41 gives too many notes. Actually the error of choosing 7/12 amounts to

$$\frac{log_2(\frac{3}{2}) - \frac{7}{12}}{log_2(\frac{3}{2})} \approx= 0.002785 \tag{88}$$

which is smaller than 0.3%.

Appendix A

Codes

The codes here are converted to Latex by C++ to Latex Web Interface (http://cpp2latex.geodar.com).

Program 1: Quadratic Equation

```
#include <iostream>
#include <iomanip>

using namespace std;

int main()
{
  long double a,b,c,x;
  cin>>a>>b>>c>>x;
  if (a!=1)
  {
    b/=a;
    c/=a;
  }
  b=-b;
  c=-c;
  for (int i=1;i<=10;i++)
  {
    long double t=x;
    for (int j=1;j<=i;j++)
      t=b+c/t;
    cout<<setprecision(18)<<i<<"&"<<t<<"\\\\"<<endl;
    x=t;
  }
  return 0;
}
```

Program 2: Gear Ratios

```
#include <iostream>
#include <iomanip>

using namespace std;

int main()
{
  long long a,b;
  cin>>a>>b;
  cout<<setprecision(12)<<a*1.0/b<<endl;
  cout<<"p_-1 = 1\tq_-1 = 0\n";
  cout<<"p_0 = "<<a/b<<"\tq_0 = 1\n";
  long long c=1,d=a/b,e=0,f=1;
  for (int i=1;i<=10;i++)
    {
      long long temp=b/(a%b),ta=temp*d+c,tb=temp*f+e;
      cout<<"p_"<<i<<" = "<<ta<<"\tq_"<<i<<" =
"<<tb<<"\t"<<ta*1.0/tb<<"\t"<<"a_"<<i<<" =
"<<temp<<"\t"<<a%b<<'/'<<b<<endl;
      e=f;
      c=d;
      d=ta;
      f=tb;
      temp=a%b;
      a=b%(a%b);
      b=temp;
    }
  return 0;
}
```

Appendix B

Some Miscellaneous Continued Fractions

This list is not restricted to simple continued fractions.

1. Lord Brouncker, about 1658

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{\dots}}}}} \tag{89}$$

2. Euler, 1737

$$e - 1 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{\dots}}}}}}} \tag{90}$$

$$e - 1 = [1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]$$

3. Stern, 1833

$$\frac{\pi}{2} = 1 - \cfrac{1}{3 - \cfrac{2 \cdot 3}{1 - \cfrac{1 \cdot 2}{3 - \cfrac{4 \cdot 5}{1 - \cfrac{3 \cdot 4}{3 - \cfrac{6 \cdot 7}{1 - \cfrac{5 \cdot 6}{3 - ...}}}}}}} \tag{91}$$

4. Lambert, 1770

$$tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - ...}}}} \tag{92}$$

5. Gauss, 1812

$$tanh(x) = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{x^2}{5 + \cfrac{x^2}{7 + ...}}}} \tag{93}$$

6. Lambert, 1770; Lagrange, 1776

For $|x| < 1$,

$$arctan(x) = \cfrac{x}{1 + \cfrac{1x^2}{3 + \cfrac{4x^2}{5 + \cfrac{9x^2}{7 + \cfrac{16x^2}{9 + ...}}}}} \tag{94}$$

7. Lagrange, 1813

For $|x| < 1$,

$$log\frac{1+x}{1-x} = \cfrac{2x}{1 - \cfrac{1x^2}{3 - \cfrac{4x^2}{5 - \cfrac{9x^2}{7 - \cfrac{16x^2}{9 - ...}}}}} \tag{95}$$

8.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{...}}}} \tag{96}$$

9.

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{...}}}} \tag{97}$$

10.

For $a^2 + b > 0$,

$$\sqrt{a^2+b} = a + \cfrac{b}{2a + \cfrac{b}{2a + \cfrac{b}{2a + \cfrac{b}{...}}}} \tag{98}$$

11. Laplace, 1805; Legendre, 1826

For $x > 0$,

$$\int_0^x e^{-u^2} du = \frac{\sqrt{\pi}}{2} - \cfrac{\frac{e^{-x^2}}{2}}{x + \cfrac{1}{2x + \cfrac{2}{x + \cfrac{3}{2x + \cfrac{4}{x + ...}}}}} \tag{99}$$

12. Lagrange, 1776

For $|x| < 1$,

$$(1+x)^k = \cfrac{1}{1 - \cfrac{kx}{1 + \cfrac{\frac{1\cdot(1+k)}{1\cdot 2}x}{1 + \cfrac{\frac{1\cdot(1-k)}{2\cdot 3}x}{1 + \cfrac{\frac{2\cdot(2+k)}{3\cdot 4}x}{1 + \cfrac{\frac{2\cdot(2-k)}{4\cdot 5}x}{1 + \cfrac{\frac{3\cdot(3+k)}{5\cdot 6}x}{1 + ...}}}}}}} \tag{100}$$

Appendix C

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