CORNELL UNIVERSITY MATHEMATICS DEPARTMENT SENIOR THESIS

A Survey of Results Regarding a Computational Approach to the Zeros of Dedekind Zeta Functions

A THESIS PRESENTED IN PARTIAL FULFILLMENT
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1 Introduction

The study of the distribution of the primes first came into focus in the late 18th century when Adrien-Marie Legendre conjectured what is now known as the Prime Number Theorem. The proof of the theorem which did not arise for another century was in itself quite enlightening. A seemingly simple statement in number theory could not be explained without the use of complex analysis.

Both Jacques Hadamard and Charles de la Vallée-Poussin proved the theorem independently in 1896. They showed:

Prime Number Theorem. Let $\Pi(x)$ be the number of prime numbers less than or equal to x. Then $\Pi(x) \sim \dfrac{x}{\log x}$. [MB06, 40]

The relation $\sim$, meaning "is asymptotic to" is an equivalence relation over the set of functions where $f \sim g$ is $\lim_{x\to\infty} f(x)/g(x) = 1$. Stated another way, the Prime Number Theorem says that the density of the primes in the integers is approximately $1/\log x$.

Despite the PNT's not yielding to easy proof, very elementary methods show that $x/\log x$ is the right order of magnitude of $\Pi(x)$. (Both Paul Erdös and Atle Selberg discovered elementary proofs of the PNT in 1949. [Erd49, Sel49])

Theorem 1.1. There exist positive constants A, B such that for sufficiently large x,

$$\frac{Ax}{\log x} \leq \Pi(x) \leq \frac{Bx}{\log x}.$$

Sketch Proof. The proof requires defining some functions related to $\Pi(x)$. First,

$$\Lambda(n) = \begin{cases} \log p, & n = p^k \text{ for some prime } p \\ 0, & \text{otherwise.} \end{cases}$$

And second,

$$T(x) = \sum_{1\leq n\leq x} \Lambda(n) \left\lfloor \frac{x}{n} \right\rfloor \tag{1}$$

where $\lfloor\cdot\rfloor$ is the greatest integer function. While defining the sum over $1 \leq n \leq x$ may seem strange, it allows for x not to be an integer which will be important in relating it to $\Pi(x)$ as $\Pi(x)$ takes an argument that need not be an integer.

First, it is important to note that

$$T(x) = \sum_{1 \le n \le x} \log n$$

which is shown as follows. Let n be an integer such that its prime factorization is $\prod_i p_i^{\alpha_i}$. (Note that $\prod$ here is a product, not the Π function.) Then

$$\begin{aligned}\Lambda(p_i)\left\lfloor \frac{x}{p_i} \right\rfloor &= \left\lfloor \frac{x}{p_i} \right\rfloor \log p_i \\ &= \log p_i^{\left\lfloor \frac{x}{p_i} \right\rfloor} \\ &= \log p_i^{\alpha_i}.\end{aligned}$$

Applying the fundamental theorem of arithmetic (the unique factorization of all integers into prime numbers) and that $\log a + \log b = \log(ab)$ gives the desired result about T.

Then, by approximating the sum by the integral of the summand:

$$T(x) = \sum_{1 \le n \le x} \log n = \int_1^x \log t dt + O(\log x) = x \log x + O(\log x). \tag{2}$$

(Note: Unless otherwise specified, $f(x) = O(g(x))$ refers to the asymptotic behavior of f as $x \to \infty$. This follows by noting that:

$$\int_n^{n+1} \log t dt - \log n = (n+1) \log \frac{n+1}{n} - 1,$$

so that:

$$\frac{1}{\log x}\left(\int_1^x \log t dt - \sum_{n=1}^{x} \log n\right) = \sum_{1 \le n \le x} \frac{(n+1)\log \frac{n+1}{n} - 1}{\log x}.$$

The result then follows by using the Taylor expansion of $\log \frac{n+1}{n} = \log(1 + \frac{1}{n}) \approx \frac{1}{n}$ and noting that allowing x to be non-integer has negligible effect.

Applying the definition of T (equation 1):

$$T(x) - 2T\left(\frac{x}{2}\right) = \sum_{1 \le n \le x} \Lambda(n)\left(\left\lfloor \frac{x}{n} \right\rfloor - 2\left\lfloor \frac{x}{2n} \right\rfloor\right), \tag{3}$$

and applying equation 2:

$$T(x) - 2T\left(\frac{x}{2}\right) = x\log 2 + O(\log x). \tag{4}$$

Then, it is important to note that for real α, $0 \leq \lfloor 2\alpha \rfloor - 2\lfloor \alpha \rfloor \leq 1$. The result is trivial if α is an integer; otherwise, let γ denote the fractional part of α (i.e., $\alpha - \lfloor \alpha \rfloor$). Then

$$\begin{aligned}
\lfloor 2\alpha \rfloor &= \lfloor 2\lfloor \alpha \rfloor + 2\gamma \rfloor \\
&\geq \lfloor 2\lfloor \alpha \rfloor \rfloor \text{ because } \lfloor \cdot \rfloor \text{ is non-decreasing} \\
&= 2\lfloor \alpha \rfloor \text{ because } \lfloor \alpha \rfloor \in \mathbb{Z}.
\end{aligned}$$

Further, noting that $\gamma < 1$ and thus $2\gamma < 2$:

$$\begin{aligned}
\lfloor 2\alpha \rfloor &= \lfloor 2\lfloor \alpha \rfloor + 2\gamma \rfloor \\
&= 2\lfloor \alpha \rfloor + \lfloor 2\gamma \rfloor \\
&\leq 2\lfloor \alpha \rfloor + 1.
\end{aligned}$$

This shows in particular that

$$0 \leq \left\lfloor \frac{x}{n} \right\rfloor - 2\left\lfloor \frac{x}{2n} \right\rfloor \leq 1.$$

Thus, equation 3 implies

$$T(x) - 2T\left(\frac{x}{2}\right) \leq \sum_{1 \leq n \leq x} \Lambda(n).$$

Further the terms for $n = p^i$ for p a prime contribute at most $\log p$ to the sum, and there are at most $\lfloor \log_p x \rfloor \leq \log_p x = \frac{\log x}{\log p}$ such primes. Then,

$$T(x) - 2T\left(\frac{x}{2}\right) \leq \sum_{p \leq x} (\log p)\frac{\log x}{\log p} = \Pi(x)\log x.$$

This gives the result (from equation 4)

$$\Pi(x) \geq \frac{x\log 2}{\log x} + O(1).$$

(Note that $\log 2 \approx 0.693$, so this is not far from the bound given in the Prime Number Theorem of $A = 1$.)

Further, taking into account the slow growth of the logarithm function, an upper bound on the number of primes between $\frac{x}{2}$ and x may be found:

$$\Pi(x) - \Pi(x/2) \leq \frac{x \log 2}{\log \frac{x}{2}} + O(1).$$

Applying this identity taking $x \leftarrow x/2$ repeatedly and summing over k_0 such inequalities gives:

$$\Pi(x) \leq 3 + 2\sum_{k=1}^{k_0} \frac{x/2^k}{\log x/2^k} + O(k_0)$$

for appropriately chosen k_0. This ultimately leads to the result that

$$\Pi(x) \leq \frac{Bx}{\log x}$$

for sufficiently large x where B is any constant larger than $\frac{20}{9}$. $\square$ [MB06, 40-43]

Using better estimates, in 1850, Pafnuty Chebyshev was able to show that

$$\frac{0.921x}{\log x} \leq \Pi(x) \leq \frac{1.105x}{\log x}$$

for sufficiently large x. [MB06, 40] This result is very close to the PNT. Though the PNT is a stronger result, it is still worth noting that Euclid proved the that there are infinitely many primes long before Hadamard and de la Vallée-Poussin proved the Prime Number Theorem.

Theorem 1.2. There are infinitely many primes.

Proof. By contradiction.

Suppose there is a finite number of primes. Let $(p_i)_{i=1}^{N}$ be an ordered enumeration of the primes; i.e., $p_i < p_{i+1}$. Then $p_N! + 1$ has a remainder of 1 when divided by any prime p_i. Thus, $p_N! + 1$ must have a prime factor larger than p_N. $\square$

While Euclid's is one of the earliest known proofs of the infinitude of the primes, proofs of this nature show-up many times throughout this study. Theorem 1.1 (and the PNT) both imply that there are infinitely many primes because $\Pi(x) > Ax/\log x$ for some $A > 0$ for sufficiently large x, and $x/\log x$ is unbounded.

Almost fifty years before Hadamard and de la Vallée-Poussin proved the Prime Number Theorem, Bernhard Riemann laid some of the important groundwork in a seemingly minor paper entitled "On the Number of Primes Less Than a Given Magnitude." The paper, a mere 8 pages, proposed one of the most famous unsolved conjectures in mathematics.

Riemann studied what most calculus students know as the $p-$series. He studied the analytic properties of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

known as the Riemann zeta function. Initially defined as a function for real $s > 1$ and trivially extended to a function for $\mathrm{Re}(s) > 1$, Riemann successfully found an analytic continuation of ζ to the entire complex plane except for a simple pole at $s = 1$. It is easy to show that $\zeta = 0$ for all values $-2m$ for m a natural number. These are known as the trivial zeros of the zeta function. Riemann further conjectured, without proof:

Riemann Hypothesis. All non-trivial zeros of the Riemann zeta function lie on the line parameterized by $\frac{1}{2} + it$, $t \in (-\infty, \infty)$. [MB06, 51,55]

Both Hadamard and de la Vallée-Poussin proved the Prime Number Theorem by showing that there are no zeros of the ζ function with real part 1 or greater. The statement of the Prime Number Theorem and the fact that there are no zeros of ζ with real part 1 or greater are, in fact, equivalent. The Riemann Hypothesis, in fact, implies better bounds on $\Pi(x)$ (see equation 5).

The study of the Riemann Hypothesis and the zeta functions was further enhanced by Richard Dedekind in the late 19th century. Dedekind explored the distribution of primes in different number fields. He extended the notion of a zeta function:

$$\zeta_K(s) = \sum_{I} \frac{1}{||I||^s}$$

where the sum is taken over all non-trivial ideals of the ring of integers of a number field K. (The Dedekind zeta function measures the properties of the ring of integers of a number field which is a Dedekind ring. For more details, see the appendix.) Note that $\zeta_{\mathbb{Q}}(s)$ is the Riemann zeta function. As will be shown, the Dedekind zeta function encodes interesting information about the distribution of prime ideals in the ring of integers of the number field K. Further, by extending the notion of the zeta function, Dedekind extended the Riemann Hypothesis to a more generalized statement that ζ_K and related L-functions have all of their zeros of the form $\frac{1}{2} + it$ or $-2m$ where t is real and m is a natural number for all number fields K.

An interesting consequence of the Riemann Hypothesis (RH) is a statement stronger than the Prime Number Theorem. The RH is equivalent to the statement:

$$\Pi(x) = \int_0^x \frac{dt}{\log t} + O(\sqrt{x}\log x) \tag{5}$$

as $x \to \infty$. [Dav00, 113] The function $\int_0^x \frac{dt}{\log t}$ is often written $\mathrm{Li}(x)$, the logarithmic integral. It has been shown empirically that a better result than this is highly unlikely as deviations in $\Pi(x)$ are at least $\mathrm{Li}(\sqrt{x}) \log\log\log x$. [Bom00, 4] The best known bounds on the error term in equation 5 is $O\left(xe^{-\frac{A(\log x)^{3/5}}{(\log\log x)^{1/5}}}\right)$ for some $A > 0$. [Wei] The logarithm of this new error term is $\log x + A(\log x)^{3/5}/(\log\log x)^{1/5}$. The logarithm of the RH error term is $\frac{1}{2}x + \log\log x$. The first term of the logarithm of the known error term is, thus, twice as large as what the RH gives, and the second term of the known error term grows very slowly compared to the RH term. For example, for $x = 10^{100}$, it is just 3.4 times larger than the RH term. These, being exponential, however, are a great deal worse than the RH bounds.

While it has been shown that all zeros of the zeta functions are either of the form $-2m$ for natural m or in the so-called "critical strip" with real values between 0 and 1, neither the General Riemann Hypothesis nor the original Riemann Hypothesis has yet been proven. Much work has, however, been done to both ends. A slew of computational work, for example, has shown that in certain ranges, the Dedekind zeta functions of certain number fields satisfy the RH (e.g., for a given number field, all zeros in the critical strip with imaginary parts between 1 and 100 are on the critical line). Computational efforts to shine light on the Riemann Hypothesis have led to results including the verification of the Riemann Hypothesis in various neighborhoods of the critical strip, notably including a neighborhood of $\frac{1}{2} + 10^{22}i$. [Odl01]

2 Early Results Regarding Zeta Functions

It is first important to note the convergence of ζ.

Theorem 2.1. For all $\mathrm{Re}(s) > 1$, ζ converges absolutely, and, in particular

$$\frac{1}{s-1} \leq \zeta(s) \leq 1 + \frac{1}{s-1}.$$

Proof. Let $f_s(x) = \frac{1}{x^s}$. Then $\zeta(s)$ approximates the integral f_s in the following way:

$$\int_1^\infty f_s(x)dx \leq \zeta(s) \leq 1 + \int_1^\infty f_s(x)dx.$$

The first inequality may be seen by viewing $\zeta(s)$ as the left-hand sum approximation of the integral of f_s from 1 to ∞, noting that f_s is decreasing. Further, the second inequality follows because $\zeta(s)$ is the right-hand sum approximation of the integral of f_s from 1 to ∞ plus the first term of 1. (The first term is problematic in the integral as $\frac{1}{x^s}$ has a pole as $x = 0$.)

Finally, the integral $\int_1^\infty f_s(x)dx = \frac{1}{s-1}$, establishing the desired result.

□ [Lan70, 157]

A simple modification of the proof above also shows:

Corollary 2.2. ζ is absolutely convergent for all complex s with $\mathrm{Re}(s) > 1$ except for a simple pole at $s = 1$ with residue 1.

Proof. The residue at $s = 1$ follows simply from the inequality in theorem 2.1, and the absolute convergence for all s with $\mathrm{Re}(s) > 1$ follows from noting that $|z^{\sigma+it}| = |z|^\sigma$ for $\sigma, t \in \mathbb{R}$. □

Further, all Dedekind zeta functions are absolutely convergent for $s > 1$. The result follows from a result on Dirichlet series, sums of the form $\sum \frac{a_n}{n^s}$.

Lemma 2.3. Let $(a_k)_k$ be an arbitrary sequence of complex numbers such that

$$\left|\sum_{k=1}^n a_k\right| = O(n^{\sigma_0})$$

for some $\sigma_0 > 0$. Then the sum

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for all s with $\mathrm{Re}(s) > \sigma_0$.

Proof. Let $m \geq n$ with both m, n large enough so that

$$|\sum_{k=1}^{m} a_k| \leq C m^{\sigma_0}$$

(respectively, the sum from 1 to n) for some $C \in \mathbb{R}$. It will be shown that the partial sums of $\sum a_n n^{-s}$ form a Cauchy sequence. For $\mathrm{Re}(s) > \sigma_0$,

$$\begin{aligned}\sum_{k=1}^{m} \frac{a_k}{k^s} - \sum_{k=1}^{n} \frac{a_k}{k^s} &= \sum_{k=1}^{m} \frac{a_k}{n^s} + \sum_{k=n+1}^{m-1} (\sum_{i=1}^{k} a_i)\Big(\frac{1}{k^s} - \frac{1}{(k+1)^s}\Big) \\ &= \sum_{k=1}^{m} \frac{a_k}{n^s} + \sum_{k=n+1}^{m-1} (\sum_{i=1}^{k} a_i) s \int_k^{k+1} \frac{dx}{x^{s+1}}.\end{aligned}$$

Let $\epsilon > 0, \mathrm{Re}(s) \geq \sigma_0 + \epsilon$. Then,

$$\Big|(\sum_{i=1}^{k} a_i) \int_k^{k+1} \frac{1}{x^{s+1}} dx\Big| \leq C \int_k^{k+1} \frac{1}{x^{\mathrm{Re}(s)-\sigma_0+1}}.$$

Thus,

$$|\sum_{k=1}^{m} \frac{a_k}{k^s} - \sum_{k=1}^{n} \frac{a_k}{k^s}| \leq C\Big(\frac{1}{n^\epsilon} + \frac{|s|}{\epsilon(m+1)^\epsilon}\Big).$$

□ [Lan70, 156-157]

Theorem 2.4. For any number field K, ζ_K is absolutely convergent for all s with $\mathrm{Re}(s) > 1$.

Proof. The number of ideals of a given order is at most $[K : \mathbb{Q}]$, so $|\zeta_K| \leq [K : \mathbb{Q}]|\zeta|$.

□ [Lon77, 123-124]

Further, it is not apparent that there should be any relationship between the Riemann zeta functions and the distribution of primes. By invoking the fundamental theorem of arithmetic, however, the result becomes clear for the Riemann zeta function.

Theorem 2.5. For all $s > 1$,

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}$$

where the product is taken over all primes p.

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^s} &= \prod_p \sum_{i=1}^{\infty} \frac{1}{(p^i)^s} \text{ by the fundamental theorem of arithmetic} \\ &= \prod_p \frac{1}{1 - p^{-s}} \text{ by simplifying the geometric series.}\end{aligned}$$

□

Note that this implies Euclid's theorem on the infinitude of primes. The fact that ζ has a pole at $s = 1$ means that there must be an infinite number of primes in order to make the product above infinite.

This theorem extends similarly to Dedekind zeta functions:

$$\zeta_K(s) = \prod_P \frac{1}{1 - ||P||^{-s}} \tag{6}$$

where the product is taken over all prime ideals P, suggesting a distribution of prime ideals in the ring of integers of a number field K similar to that of the distribution of the primes in the integers. (For short-hand, the ideals of the ring of integers of a number field will be abbreviated as the ideals of the number field.) Note that this theorem makes use of the multiplicativity of $|| \cdot ||$. Further, note that this theorem is important in that Dedekind rings are not generally unique factorization domains; that is, elements of the rings do not factor uniquely. Instead, ideals factor uniquely, providing the analog for primes in the integers (which are in a one-to-one, natural correspondence with the prime ideals).

Another important property of the Riemann zeta function is its analytic continuation (except at $s = 1$). The continuation to $\mathrm{Re}(s) > 0$ involves a simple trick.

Theorem 2.6. ζ may be extended to a meromorphic function for all $\mathrm{Re}(s) > 0$ with a simple pole at $s = 1$.

Proof. Let $\zeta_2(s) = \sum_{n=1}^{\infty}(-1)^{n+1}n^{-s}$ which converges (conditionally) for $\mathrm{Re}(s) > 0$ by lemma 2.3 (as the partial sums of the numerator are either 1 or 0).

Then

$$\zeta(s) = \zeta_2(s) + \frac{2}{2^s}\zeta(s), \text{ so}$$
$$\zeta(s) = \frac{\zeta_2(s)}{1 - 2^{1-s}}$$

which provides a meromorphic extension except for possible poles at $1 + \frac{2\pi i n}{\log 2}$ for $n \in \mathbb{Z}$. A similar construction,

$$\zeta_3(s) = \frac{1}{1^s} + \frac{1}{2^s} - \frac{2}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} - \frac{2}{6^s} + \cdots$$

shows, however, that the poles may only occur where $2^n = 3^m$ which is impossible as $(2, 3) = 1$. □ [Mar77, 184-186]

Finally, ζ may be extended analytically to all $s \in \mathbb{C}$ except for $s = 1$. A functional equation for ζ relates $\zeta(s)$ to $\zeta(1 - s)$. (Note that in the functional equation below, Λ is not the same as the Λ function used in the proof of theorem 1.1. The use of the letter Λ is maintained as it is standard notation for both.)

Functional Equation. Let $\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. Then $\Lambda(s) = \Lambda(1 - s)$, and $s(1 - s)\Lambda(s)$ is analytic on the entire complex plane.

Proof. In order to prove the functional equation of ζ, a function with a known functional equation will be used. The function

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$$

has an easy to verify functional equation:

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right).$$

This follows directly from the fact that $e^{-\pi x^2}$ is its own Fourier transform and the Poisson summation formula. This is the functional equation for θ. Further, let

$$\psi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

so that $\theta(t) = 1 + 2\psi(t)$.

Then using that $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$, evaluating at $s/2$ and using the change of variables $t \leftarrow n^2\pi t$,

$$\pi^{-s/2}\Gamma(s/2)n^{-s} = \int_0^\infty e^{-\pi n^2 t}t^{s/2-1}dt.$$

Then, summing over n from 1 to infinity,

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty \psi(t)t^{s/2-1}dt$$

for $\mathrm{Re}(s) > 1$. Note that the integral and sum may be interchanged as both are absolutely convergent for $\mathrm{Re}(s) > 1$. Noting that this is the expression for $\Lambda(s)$, and substituting for θ:

$$\begin{aligned}\Lambda(s) &= \int_1^\infty \psi(t)t^{s/2}\frac{dt}{t} + \frac{1}{2}\int_0^1 \theta(t)t^{s/2}\frac{dt}{t} - \frac{1}{2}\int_0^1 t^{s/2}\frac{dt}{t} \\ &= \int_1^\infty \psi(t)t^{s/2}\frac{dt}{t} + \frac{1}{2}\int_0^1 \theta(t)t^{s/2}\frac{dt}{t} - \frac{1}{s}.\end{aligned}$$

Then, using the change of variables $t \leftarrow \frac{1}{t}$ and applying the functional equation for θ:

$$\begin{aligned}\frac{1}{2}\int_0^1 \theta(t)t^{s/2}\frac{dt}{t} &= \frac{1}{2}\int_1^\infty \theta(\frac{1}{t})t^{-s/2}\frac{dt}{t} \\ &= \frac{1}{2}\int_1^\infty \theta(t)t^{\frac{1-s}{2}}\frac{dt}{t} \\ &= \int_1^\infty \psi(t)t^{\frac{1-s}{2}}\frac{dt}{t} + \frac{1}{2}\int_1^\infty t^{\frac{1-s}{2}}\frac{dt}{t} \\ &= \int_1^\infty \psi(t)t^{\frac{1-s}{2}}\frac{dt}{t} - \frac{1}{1-s}.\end{aligned}$$

Thus,

$$\Lambda(s) = \int_1^\infty \psi(t)\big(t^{\frac{s}{2}} + t^{\frac{1-s}{2}}\big)\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}$$

which is clearly invariant under $s \leftarrow 1-s$ and analytic when multiplied by $s(1-s)$.

□ [Bum]

The functional equation has several important consequences. First, noting that Γ has a pole at $-n$, $\zeta(-2n)$ must be 0 to satisfy the functional equation as $\Lambda(-2n)$ is finite. Second, Γ has a simple pole at $s = 0$, so $\zeta(0)$ is not a pole. (If it were, then the pole of ζ at $s = 1$ would not be simple.) Further, it implies that all of the other zeros of ζ are in the critical strip $0 \le \mathrm{Re}(s) \le 1$. There are no zeros for $\mathrm{Re}(s) > 1$ by the product formula for ζ; thus, there are no zeros other than those $-2m$ in the regions $\mathrm{Re}(s) > 1, \mathrm{Re}(s) < 0$. Further, the zeros in the critical strip are distributed symmetrically about $\mathrm{Re}(s) = \frac{1}{2}$. It also leads to the proof of the PNT (in its equivalent form about the zeros of ζ).

Theorem 2.7. There are no zeros of ζ on the line $\mathrm{Re}(s) = 1$.

Sketch Proof. From theorem 2.5,

$$\log\zeta(s) = \sum_p \sum_{m=1}^\infty m^{-1}p^{-ms}.$$

Equivalently, letting $\sigma = \mathrm{Re}(s), t = \mathrm{Im}(s)$,

$$\log\zeta(s) = \sum_p \sum_{m=1}^\infty m^{-1}p^{-m\sigma}e^{-m\log t}.$$

If $\zeta(1+it)$ were zero for some real t, then $\mathrm{Re}(\log\zeta(1+it))$ would tend to $-\infty$ as $\sigma \to 1^+$. This would imply that the real part of the sum,

$$\mathrm{Re}\log\zeta(s) = \sum_p \sum_{m=1}^\infty m^{-1}p^{-ms}\cos(tm\log p) \tag{7}$$

would be dominated by negative terms. However, this would seem to imply that $\cos(2tm\log p)$ should also be dominated by negative values. If that were true, then

$\operatorname{Re}\log\zeta(\sigma+2it)$ would also tend to $-\infty$ as $\sigma \to 1^+$. It will be shown that this is a contradiction.

The result follows from a trigonometric identity. Applying the fact that $\cos 2\theta = 2\cos^2\theta - 1$, it can be shown that:

$$0 \leq 2(1+\cos\theta)^2 = 3 + 4\cos\theta + \cos 2\theta.$$

Then, taking $\theta = t\log p^m$ gives

$$3\cos 0 + 4\cos(t\log p^m) + \cos(2t\log p^m) \geq 0.$$

Multiplying each term by $m^{-1}p^{-m\sigma}$ and summing over p, m gives:

$$3\log\zeta(\sigma) + 4\operatorname{Re}\log(\sigma+it) + \operatorname{Re}\log\zeta(\sigma+2it) \geq 0$$

which gives (by exponentiation and noting that the real $e^{\operatorname{Re}(z)} = |z|$)

$$\zeta^3(\sigma)\left|\zeta^4(\sigma+it)\zeta(\sigma+2it)\right| \geq 1 \tag{8}$$

for $\sigma > 1$. (Note that it is necessary that $\sigma > 1$ as the product formula for ζ only applies in this case.)

Further, because $s = 1$ is a simple pole of ζ, $\zeta(s) = \frac{1}{s-1} + o(\frac{1}{s-1})$ as $s \to 1$. However, if $\zeta(1+it) = 0$ for some $t \neq 0$, then

$$|\zeta(\sigma+it)| < A(\sigma-1)$$

for some A as $\sigma \to 1$ from the right. However, it can be shown that $\zeta(\sigma+2it)$ is bounded, giving a contradiction with inequality 8. This bound on $\zeta(\sigma+2it)$ follows from finding a summation formula similar to equation 7 for $\operatorname{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right)$.

□ [Dav00, 84-87]

Corollary 2.8. There are no zeros of ζ on the line $\operatorname{Re}(s) = 0$.

Proof. This follows directly from the functional equation and theorem 2.7.

The Dedekind zeta function, further, satisfies a functional equation similar to that of the Riemann zeta function. (For definitions of these terms, see the appendix.)

Functional Equation. Let r_1 denote the number of real embeddings of K, and let r_2 denote the number of conjugate pairs of complex embeddings of K. Then let

$$\Lambda_K(s) = \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \left(\frac{\sqrt{|\mathrm{disc}(K)|}}{\pi^{\frac{[K:\mathbb{Q}]}{2}} 2^{r_2}}\right)^s \zeta_K(s)$$

Then

$$\Lambda_K(s) = \Lambda_K(1-s).$$

Part of what makes the Generalized Riemann Hypothesis and the study of the Dedekind zeta functions so important is the relationship between a complex function and inherent properties of number fields. In addition to the discriminant and the index of the number field in $\mathbb{Q}$, the Dedekind zeta function also contains information about the regulator of K and the class number. In particular the residue of the pole at $s = 1$ combines many important constants:

Theorem 2.9. Let h_K denote the class number of K. Let r_1 and r_2 be the number of real embeddings and complex conjugate pairs of embeddings of K, respectively. Let $\mathrm{Reg}(K)$ be the regulator of K. And let w be the number of roots of unity in K. Then

$$\lim_{s\to 1}(s-1)\zeta_K(s) = h_k \frac{2^{r_1}(2\pi)^{r_2}\mathrm{Reg}(K)}{w\sqrt{|\mathrm{disc}(K)|}}.\text{[Lan70, 161]}$$

The study of so-called L-functions allows for further analysis of Dedekind zeta functions as well. Let $G = (\mathbb{Z}/m\mathbb{Z})^*$, the multiplicative group of $\mathbb{Z}/m\mathbb{Z}$. Then χ is a character mod m if it is a multiplicative homomorphism $G \to \mathbb{C}^*$. Let f denote the quotient homomorphism $\mathbb{Z} \to G$. Then m is called the conductor of χ if χ is not induced by a homomorphism $G \to (\mathbb{Z}/l\mathbb{Z})^*$; that is, if there is no proper divisor l of m so that χ is also a character mod l. In this case, χ is called primitive. Then, by slight abuse of notation, taking $\chi(n) := \chi(f(n))$ if $(m, n) = 1$ and 0 otherwise, the L-functions are defined for $\mathrm{Re}(s) > 1$

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Note that the sum may be taken over n with $(m, n) = 1$ as $\chi(n) = 0$ if $(m, n) \neq 1$. The characters χ which are everywhere 1 are generally referred to as the trivial characters. Further, most L-functions can trivially be extended to functions that converge for $\mathrm{Re}(s) > 0$

Theorem 2.10. For χ non-trivial, the sum $L(s, \chi)$ converges for all $\mathrm{Re}(s) > 0$.

Proof. Let $g_0 \in G$ so that $\chi(g_0) \neq 1$. (Note that here it is being used that χ is non-trivial; trivial characters are always 1.) Then the sum

$$\sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(g_0 g) = \sum_{g \in G} \chi(g_0)\chi(g) = \chi(g_0) \sum_{g \in G} \chi(g).$$

Thus, the sum must be zero, so the partial sums are $O(1)$, so lemma 2.3 implies the desired result. □

Further, if χ is trivial, then $L(s, 1) = \zeta(s)$. In addition to this result, there is a functional equation for each L-function that gives an analytic continuation to the entire complex plane. Also, in a manner similar to the product formulas for ζ and ζ_K, $L(s, \chi)$ may be written as the product over all primes p:

$$L(s, \chi) = \prod_p (1 - \chi(p) p^{-s})^{-1} \tag{9}$$

for $\mathrm{Re}(s) > 1$. Further, the main result on L-functions and ζ_K allows, for certain K, a decomposition of ζ_K into a product of L-functions.

Theorem 2.11. Let K be the field generated by a primitive n^{th} root of unity. Then

$$\zeta_K(s) = \prod_\chi L(s, \chi)$$

where the product is taken over χ with conductors dividing n.

Proof. Note that because ζ_K and L both have analytic extensions (except, in the cases of ζ_K and $L(s, 1)$ for a simple pole at $s = 1$), the result will follow if it is shown for the defining regions of the functions, $\mathrm{Re}(s) > 1$.

Note that by applying the product formulas in equations 6, 9 and taking the reciprocal of both sides, the result is equivalent to

$$\prod_{P|(p)} (1 - ||P||^{-s}) = \prod_\chi (1 - \chi(p) p^{-s}) \tag{10}$$

where the product is taken over all prime ideals lying under (p).

Let $r \geq 0, m \geq 1$ be integers so that $n = p^r m$ and $(m,p) = 1$. Then let $\{P_1, P_2, \ldots P_g\}$ be the set of prime ideals in the ring of integers of K lying under (p) so that $(P_1 P_2 \cdots P_g)^e = (p)$ where $e = \phi(p^r)$, the number of divisors of p^r. (ϕ is often called the Euler totient function.) Then let f be the order of $p \mod m$; that is, $||P_i|| = p^f$. Finally, then e, f, g must satisfy $efg = \phi(n)$ because n is the degree of K. This leads to the result that the left-hand side of equation 10 is

$$\prod_{i=1}^{g}(1 - ||P_i||^{-s}) = \prod_{i=1}^{g}(1 - p^{-fs}) = (1 - p^{-fs})^g.$$

Now it will be shown that the right-hand-side of 10 is equal to this. Let χ be a character whose conductor divides n. Let f_χ denote the conductor of χ. Then, if $f_\chi \not| m$, then $p | f_\chi$. (This follows from p being prime and dividing n.) In this case, $\chi(p) = 0$ so that the right-hand side of equation 10 may be written

$$\prod_{f_\chi | m}(1 - \chi(p)p^{-s}). \tag{11}$$

Further, for all χ such that $f_\chi | m$, there is some χ' on $(\mathbb{Z}/m\mathbb{Z})^*$ that is induced by the quotient homomorphism $(\mathbb{Z}/m\mathbb{Z})^* \to (\mathbb{Z}/f_\chi\mathbb{Z})^*$. As χ runs through $f_\chi | m$, χ' runs through the dual of $(\mathbb{Z}/m\mathbb{Z})^*$. (An important result in the theory of characters is that all χ' on $(\mathbb{Z}/m\mathbb{Z})^*$ are induced by a unique primitive χ.)

Then if $z \in (\mathbb{Z}/m\mathbb{Z})^*$ is congruent to $p \mod m$ then $\chi(p) = \chi'(z)$ (because χ' is induced by χ). Further, since the order of z is f (by definition), $\chi'(z)$ is an f^{th} root of unity. Then as χ' runs through all characters on $(\mathbb{Z}/m\mathbb{Z})^*$, $\chi'(z)$ runs through the group of f^{th} roots of unity. In particular, $\chi'(z)$ takes the value of each f^{th} root of unity $\frac{\phi(m)}{f} = g$ times as there are $\phi(m)$ elements of $(\mathbb{Z}/m\mathbb{Z})^*$.

This finally allows equation 11 to be reduced to the product over the f^{th} roots of unity:

$$\prod_{\omega^f = 1}(1 - \omega p^{-s})^{-g} = (1 - p^{-fs})^g.$$

□ [Lon77, 126-127]

More generally, as can be shown by extending the proof above,

Theorem 2.12. Let $G = \mathrm{Aut}(K/\mathbb{Q})$. If G is abelian then

$$\zeta_K(s) = \prod_{\chi} L(s,\chi)$$

where the product is taken over all χ where the conductor of χ divides the discriminant of K. [Lon77, 129], [IK04, 126]

Note, in particular, that if K is a quadratic extensions, $\mathrm{Aut}(K/\mathbb{Q})$ is abelian.

3 Zeros of the Zeta Functions

Attempts to prove the Riemann Hypothesis have resulted in much study of the zeta function. For example, Riemann himself outlined a proof that

Theorem 3.1. Let $N(T)$ be the number of zeros in the with imaginary part between 0 and T and real part at least $\frac{1}{2}$. Then

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} + O(\log T).$$

Sketch Proof. First, it is useful to define the function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}s(s-1)\Lambda(s).$$

The function $\xi(s)$ has all of the zeros of ζ with added zeros at $s = 0, 1$. The advantage to using $\Lambda(s)$ over $\zeta(s)$ is its symmetry, and the advantage to using $\xi(s)$ over $\Lambda(s)$ is that is has no poles (as shown in the proof of the functional equation) and satisfies the symmetry conditions:

$$\xi(\sigma+it) = \xi(1-\sigma-it); \xi(\sigma+it) = \overline{\xi(\sigma-it)} \tag{12}$$

for σ, t real. The first statement of symmetry is the same as the functional equation, and the second basically says that ξ admits conjugates. Further it is important to note that since $s\Gamma(s) = \Gamma(s+1)$,

$$\xi(s) = (s-1)\pi^{-s/2}\Gamma(\frac{s}{2}+1)\zeta(s). \tag{13}$$

Without loss of generality, assume T is not a zero of ξ. (If T is a zero then taking $N(T+\delta)$ or $N(T-\delta)$ for some $\delta > 0$ should give the desired value of N. It is desirable that T not be a zero so that the argument principle or Cauchy integral formula may be applied.) Then let R be the rectangle with vertices $\{2, 2+iT, -1+iT, -1\}$ oriented counter-clockwise; that is, R is the boundary of the convex hull of these points.

Then

$$2\pi N(T) = \Delta_R \arg \xi(s)$$

where Δ_R denotes the change along R. Note that the Cauchy integral formula implies the so-called "argument principle" that for a complex function f, $\Delta_R \arg f(s)$ is equal to the sum of the residues of the simple poles of f enclosed in R.

Noting the symmetries given in equation 12, the change from $\frac{1}{2} + iT$ to $-1 + iT$ to -1 is the same as the change from 2 to $2 + iT$ to $\frac{1}{2} + iT$. Thus

$$\pi N(T) = \Delta_L \arg \xi(s)$$

where L is the set of points on the lines connecting 2 to $2 + iT$ and then $2 + iT$ to $\frac{1}{2} + iT$.

Before evaluating $N(T)$, it is important to note a few results. The first, regarding the $\Gamma-$function, is known as Stirling's formula which says that as $|s| \to \infty$,

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + O(|s|^{-1}). \tag{14}$$

(Note that this is analogous to Stirling's formula for factorials: $n! = \sqrt{2\pi n}(n/e)^n(1 + O(\frac{1}{n}))$.) Further, for real T,

$$\tan^{-1} 2T = \frac{\pi}{2} + O(T^{-1})$$

as shown:

Note that this is equivalent to showing that:

$$\lim_{T\to\infty} \left| T \left(\tan^{-1} 2T - \frac{\pi}{2} \right) \right|$$

is bounded. This is shown by applying L'Höpital's rule:

$$\lim_{T\to\infty} \frac{\tan^{-1} 2T - \frac{\pi}{2}}{\frac{1}{T}} = \lim_{T\to\infty} \frac{\frac{2}{1+4T^2}}{-\frac{1}{T^2}} = -\frac{1}{2}.$$

Then letting $\sigma = \mathrm{Re}(s), t = \mathrm{Im}(s)$ and evaluating the change in argument of each factor of ξ from equation 13:

$$\begin{aligned}\Delta_L \arg(s-1) &= \tan^{-1}(2T) = \frac{1}{2}\pi + O(T^{-1}) \\ \Delta_L \arg \pi^{-\frac{1}{2}s} &= \Delta_L \arg \pi^{-\frac{1}{2}\sigma} e^{-i\frac{t}{2}\log \pi} \\ &= \Delta_L \left(-\frac{t}{2}\log \pi\right) = -\frac{T}{2}\log \pi\end{aligned}$$

Further, for the Γ function, applying equation 14,

$$\begin{aligned}\Delta_L \arg \Gamma\left(\frac{1}{2}s+1\right) &= \arg \Gamma\left(\frac{1}{4}+i\frac{T}{2}+1\right) - \arg \Gamma(2) \\ &= \arg \Gamma\left(\frac{5}{4}+i\frac{T}{2}\right) \\ &= \mathrm{Im} \log \Gamma\left(\frac{5}{4}+i\frac{T}{2}\right) \\ &= \mathrm{Im}\left(\left(\frac{3}{4}+i\frac{T}{2}\right)\log\left(\frac{5}{4}+i\frac{T}{2}\right) - \frac{1}{2} - \frac{5}{4} + \frac{1}{2}\log 2\pi + O(T^{-1})\right) \\ &= \frac{1}{2}T\log\frac{1}{2}T - \frac{1}{2}T + \frac{3}{8}\pi + O(T^{-1}).\end{aligned}$$

Then collecting all of the argument changes above and using equation **??**:

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + \Delta_L \arg \zeta(s) + O(T^{-1}).$$

The remainder of the proof requires showing that $\Delta_L \arg \zeta(s) = O(\log T)$. The crux of this part of the proof is bounding the sum $(1+(T-\gamma)^2)^{-1}$ where γ runs over the imaginary parts of the non-trivial zeros of ζ. This sum is, in fact, $O(\log T)$ which ultimately implies $\Delta_L \arg \zeta(s)$ is $O(\log T)$ also, completing the proof.□ [Dav00, 73, 97-100]

Further, it has been shown that the zeros of the Dedekind zeta functions satisfy:

Theorem 3.2. Let $N_K(T)$ denote the number of zeros of ζ_K in the critical strip. Then

$$N_K(T) \sim \frac{[K:\mathbb{Q}]}{\pi} T \log T. \quad \text{[HB77, 169]}$$

This theorem leads immediately to the result:

Corollary 3.3. Let $N'_K(T)$ denote the number of zeros on the critical line. Then

$$N'_K(T) = O(T \log T).$$

And trivially, if the Riemann Hypothesis is true, then $N'_K(T) \sim \frac{[K:\mathbb{Q}]}{\pi} T \log T$, but the upper bound has not been shown unconditionally. Letting $N_K(\sigma, T)$ denote the number of zeros of $\zeta_K(s)$ with $\mathrm{Im}(s) \geq \sigma$, the best unconditional bound is that given $\epsilon > 0$, there exists C so that

$$N_K(\sigma, T) = O(T^{([K:\mathbb{Q}]+\epsilon)(1-\sigma)} \log^C T). \text{ [HB77]}$$

An interesting question remains, however, what the pattern of the low zeros of the Dedekind zeta function is. It has been shown that

Theorem 3.4. Fix the degree of K. Then for all T, there exists some $c(T) > 0$ so that

$$N_K(T) = c(T) \log |\mathrm{disc}(K)| + o(|\mathrm{disc}(K)|). \text{ [Odl01, 11]}$$

It is important to note that this result is *not* a statement on the asymptotic behavior of N_K as $T \to \infty$, but rather, as $|\mathrm{disc}(K)| \to \infty$. In particular, this result gives the behavior of the small zeros of ζ_K.

Note that the functional equation shows that the zeros in the critical strip are distributed symmetrically about $\mathrm{Im}(s) = 0$. Further, the product formula given in theorem 2.11 shows that ζ_K for quadratic K shares all of the zeros of the Riemann zeta function. The first zero of the Riemann zeta function occurs near $s = \frac{1}{2} + 14.135$, and, thus, the first zero of each quadratic Dedekind zeta function can have imaginary part no larger than 14.135. It has, in fact, been shown that the first zero of every Dedekind zeta function has imaginary part less than 14. It has further been shown that the first zero of the Dedekind zeta function is at height $0.54 + o(1)$ as the degree of K increases without bound. [Odl89, 11] Let $t_0(K)$ denote the first zero of the Dedekind zeta function for the field K. Then, the following result is conditional:

Theorem 3.5. Assuming GRH,

$$t_0(K) = O\left(\frac{1}{\log^2 |[K:\mathbb{Q}]|}\right). \text{[Oma02, 1 – 2]}$$

Further, for a fixed degree, it has been shown conditionally

Theorem 3.6. Assuming GRH,

$$t_0(K) = O\left(\frac{1}{\log\log|\mathrm{disc}(K)|}\right).\text{[Oma00]}$$

Both of these results follow from an identity due to André Weil.

Theorem 3.7. Let F be a function from the reals to the reals satisfying the two conditions:

1) F is continuously differentiable except at a finite number of points where it has a discontinuity of the first kind; i.e., $F(x_0) = \frac{1}{2}\left(\lim_{x\to x_0^+} F(x) + \lim_{x\to x_0^-} F(x)\right)$ where x_0 is a point of discontinuity.
2) There exists $b > 0$ such that both $F(x)$ and $F'(x)$ are $O\left(e^{-\left(\frac{1}{2}+b\right)|x|}\right)$ as $|x| \to \infty$.

Then let

$$\Phi(s) = \int_{-\infty}^{\infty} F(x)e^{(s-\frac{1}{2})x}dx.$$

Further, let $n = [K : \mathbb{Q}]$, and define the functions

$$J(F) = \int_0^{\infty} \frac{F(x)}{\cosh\frac{x}{2}}dx, I(F) = \int_0^{\infty}\left(\frac{F(x)}{2\sinh\frac{x}{2}} - \frac{e^{-x}}{x}\right)dx.$$

Then summing over the non-trivial zeros $\rho = \beta + i\gamma$ of ζ_K:

$$\sum_{\rho}\Phi(\rho) = \Phi(0) + \Phi(1) - 2\sum_{P,m}\frac{\log||P||}{||P||^{m/2}}F(m\ln||P||) + F(0)\left(\log|\mathrm{disc}(K)| - n\log(2\pi)\right) - r_1 J(F) - nI(F).\text{[Oma01, 1608]}$$

The sum on the right-hand-side is taken over all prime ideals P and all natural numbers m.

The result of theorem 3.6, for example, can be found by taking

$$F(x) = \begin{cases}(1-x)\cos\pi x + \frac{3}{\pi}\sin\pi x, & x\in[0,1]\\ 0, & \text{otherwise.}\end{cases}\ \text{[Oma00, 63]}$$

Further, empirically, it seems that an ever better result is possible:

Conjecture 3.8. Fixing the signature of K (that is, the numbers r_1, r_2 which correspond to the number of real, complex embeddings of K, respectively),

$$t_0(K) = O\left(\frac{1}{\log|\mathrm{disc}(K)|}\right).[\text{Tol97}, 1318]$$

The fact that the signature is fixed means that the implied constant depends on (r_1, r_2).

An interesting follow-up question to the location of the first zero is what happens to the value of $\zeta_K(\frac{1}{2})$. It seems, at first glance, that $\zeta_K(\frac{1}{2})$ might tend to 0 as $|\mathrm{disc}(K)| \to \infty$. However, this is not implied by any of the above results. It is easy to construct a function on a line where the first zero of the function tends to zero, but the value of the function at 0 is unbounded. Consider the sequence of functions l_n which are lines connecting the points $(\frac{1}{n}, 0)$ and $(0, n)$. Then, the first (only) zero of l_n tends to zero as $n \to \infty$, but $l_n(0) \to \infty$ as $n \to \infty$. The value of $\zeta_K(\frac{1}{2})$ is explored more in section 4.

Several factors suggest that t_0 should decrease as $|\mathrm{disc}(K)|$ increases. First, a result akin to theorem 2.11 actually applies to all fields where $\mathrm{Aut}(K)$ is abelian. For these fields, as the degree of the field extension increases, there are more L-functions in the product. When any of these functions has a zero, ζ_K has a zero. Further, although theorem 3.2 is a statement about the asymptotic density of zeros, it says that the density of the zeros of the zeta functions as $|\mathrm{disc}(K)|$ increases.

3.1 Methods

The first set of data examines what happens to $t_0(K)$ as $\mathrm{disc}(K)$ increases.

GP/Pari was used which uses an algorithm due to Emmanuel Tollis. [Tol97] In order to compute ζ_K, Tollis's algorithm first computes several invariants of the field K related to ζ_K but not dependent on the choice of s. These invariants include all of the values in theorem 2.9.

Utilizing the GP/Pari algorithm, zeros were found by starting from $s = \frac{1}{2} + it, t = 0$ and moving up the imaginary axis in increments of $\Delta t = \frac{1}{10}$. Let $\tau(t)$ denote the zero given by 3 iterations of Newton's Method with a starting point of t. Then $\tau(t)$ was computed and rejected unless $|t - \tau(t)| < \frac{1}{10}$ in an effort to find the lowest zero. Once a suitable $\tau(t)$ was found, five more iterations of Newton's Method were run to refine the estimate. Finally, to ensure that this was, in fact, the lowest zero of ζ_K, the integral

$$\frac{1}{2\pi i}\int_{\gamma(\tau(t))} \frac{\zeta_K'(s)}{\zeta_K(s)} ds \tag{15}$$

was computed. Here, $\gamma(\tau(t))$ is a circle about the point $s = \frac{1}{2}$ of radius $\tau(t)$. This gives the number of zeros contained in the circle by the residue theorem. Thus, the first zero could be verified by ensuring that the this integral minus the trivial zeros contained in γ (where applicable) and the values of the poles at $s = 0, 1$ is equal to 1. Note that while it may seem more computationally efficient to integrate to integrate around $\gamma(\tau(t))$ after the first three iterations of Newton's Method, in practice, it is more time consuming to compute the integral than a few iterations of Newton's Method. Additionally, the first zeros computed by Newton's Method (for small t) were often not the lowest zeros.

Further, in an attempt to test the relationship given in theorem 3.4, data were taken for the number of zeros between imaginary parts 0 and the values $2\log 2$, $3\log 3$, $4\log 4$, $5\log 5$, $6\log 6$, $7\log 7$, and $8\log 8$ (approximately 1.39, 3.30, 5.55, 8.05, 10.75, 13.62, and 16.64, respectively) were taken. These were computed using the integral 15 with radii as given (in place of $\tau(t)$). Note that these values were used to get a distribution of values in a range of small numbers; there is no special significance of the values $n\log n$.

The values given for the number zeros is generally not integral due to rounding errors in the computation of the integral. Additionally, though it may seem more computationally efficient not to compute the integral around such a large circle and, instead, compute the integrals about several circles and add them, rounding error makes this difficult. In particular, when the circle of integration is near a zero of ζ_K, the rounding may cause errors that are quite large.

Data were collected using fields of the form $\mathbb{Q}[\sqrt[m]{d}]$ (a degree m extension of $\mathbb{Q}$) with $d = 2^n + 1$. Statistical data and graphs were created using gnuplot. All best-fit curves were computed as best-fit lines; for example, for $N_K(T) = c(T)\log|\mathrm{disc}(K)|$,

T	$\overline{c_2}(T)$	Error
1.386	0.172	0.0150
3.296	0.432	0.0225
5.545	0.809	0.0214
8.047	1.217	0.0227
10.751	1.636	0.0132
13.621	2.155	0.0200
16.636	2.579	0.0361

Table 1: Values of $c_2(T)$ (i.e., $c(T)$ for quadratic fields).

a plot was made of $\log|\mathrm{disc}(K)|$ versus $N_K(T)$.

3.2 Asymptotic Behavior of $N_K(T)$ as $|\mathrm{disc}(K)|$ Grows

As per theorem 3.4, the relationship $N_K(T) = c(T)\log|\mathrm{disc}(K)|$ was tested first. The data for $c(T)$ are quite suggestive. See, for example, figure 1. In this section, $c_n(T)$ will refer to $c(T)$ for fields of degree n. (Note that in this section, only the degree of K matters; the signature does not.) For quadratic and cubic fields, and for all values of T, the value $c_n(T)$ was found with a relatively high accuracy. The data presented in tables 3.2 and 3.2 give the values of T, $\overline{c_n}(T)$, and an error value σ. The last value corresponds to 2 standard errors, giving a probability of 95.5% that the actual value of $c_2(T)$ is in the range $[\overline{c}(T) - \sigma, \overline{c}(T) + \sigma]$.

Plotting values of $c(T)$ vs. T gives fairly compelling evidence that the relationship is simply linear. For quadratic fields, the relationship

$$c_2(T) = 0.154T$$

works fairly well as shown in figure 2. The coefficient 0.154 has a standard error of less than 0.002, meaning that there is a greater than 95.5% chance that the coefficient is between 0.150 and 0.158. Combining these results,

T	$\overline{c_3}(T)$	Error
1.386	0.129	0.0294
3.296	0.424	0.0206
5.545	0.747	0.0292
8.047	1.187	0.0426
10.751	1.651	0.0388
13.621	2.142	0.0487
16.636	2.605	0.0687

Table 2: Values of $c_3(T)$ (cubic fields).

Observed Relationship 3.9. For quadratic K and small T (less than 20),

$$N_K(T) = 0.154T \log |\mathrm{disc}(K)|.$$

Further, the coefficient for cubic K is 0.154 ± 0.005. (Here, $\pm$ denotes twice the standard error, corresponding to the same confidence intervals as earlier.)

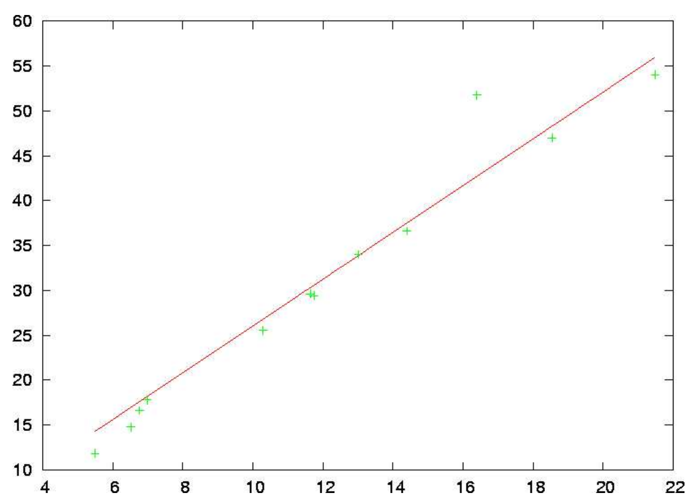

Figure 1: $N_K(8 \log 8)$ vs. $\log |\mathrm{disc}(K)|$ for cubic fields.

The result in observation 3.9 is quite interesting in that is says that the asymptotic behavior of the zeros of ζ_K is different from that of small zeros. In particular, as

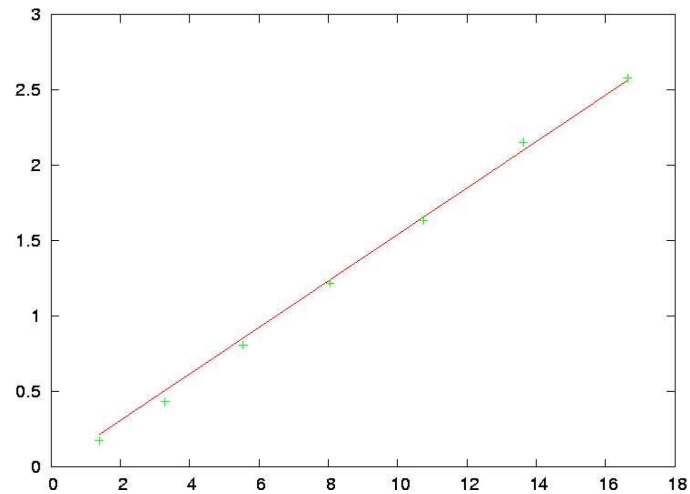

Figure 2: $c_2(T)$ vs. T with best-fit line $c_2(T) = 0.154T$.

$T \to \infty$, $N_K(T) \propto T \log T$ as per theorem 3.2, yet they are infinitely less dense in the regions observed. For small T, $N_K(T) \propto T$.

3.3 Asymptotic Behavior of $t_0(K)$ as $|\mathrm{disc}(K)|$ Grows

Despite conjecture 3.8 being based only on computational evidence, the evidence found here not only reinforces the work in [Tol97], but makes it seem that the relationship proposed therein is quite natural. That is, it seems that one may not be able to improve upon the result

$$t_0(K) = O\left(\frac{1}{\log|\mathrm{disc}(K)|}\right)$$

as the variations from $\frac{C}{\log|\mathrm{disc}(K)|}$ for appropriate C are small.

For quadratic fields K with signature $(1, 0)$, the best-fit line for t_0 vs. $\frac{1}{\log|\mathrm{disc}(K)|}$ has a slope of 0.961 ± 0.064. This relationship is shown in figure 3. The fact that the best-fit line has a slope of almost 1 and that its standard error is so low indicates that this may be close to the best bound possible. In particular, if the coefficient had a large standard error, it would be reasonable to assume either that t_0 is not bounded above by a multiple of $(\log|\mathrm{disc}(K)|)^{-1}$ or that there is a smaller bounding function. For example, attempting to fit a curve of the form $(\log\log|\mathrm{disc}(K)|)^{-1}$ as per theorem 3.6 yields a significantly larger standard error (in relative terms, more than twice as large).

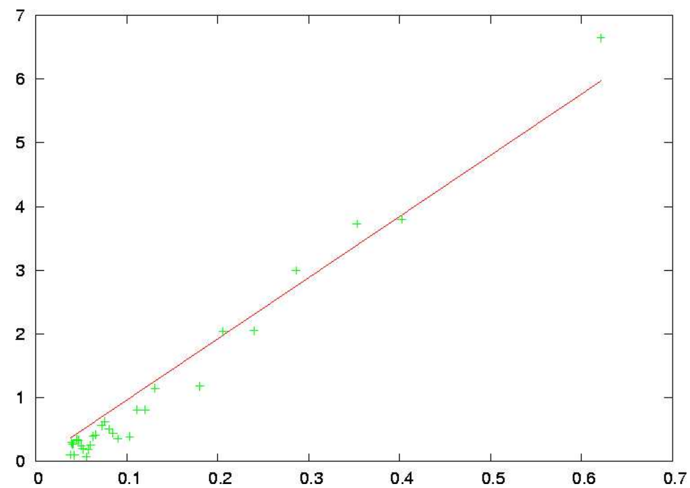

Figure 3: t_0 vs. $\frac{1}{\log|\mathrm{disc}(K)|}$ for quadratic fields. The best-fit line $t_0 = \frac{0.961}{\log|\mathrm{disc}(K)|}$ is included.

Note that cubic fields of the form $\mathbb{Q}[\sqrt[3]{a}]$ for real a have signature $(1,1)$ as $\sqrt[3]{a}$ may be mapped to itself or to a $\sqrt[3]{a}e^{2\pi i/3}$ or $\sqrt[3]{a}e^{4\pi i/3}$. Thus all cubic fields used here have signature $(1,1)$. For cubic fields, the same relationship was found; here the coefficient was found to be 10.942 ± 0.649. This relationship is reflected in figure 4. For sixth-degree fields, best-fit straight line has a slope of 21.277 ± 2.912. (Note that the larger error here is due to there being less data as the discriminant of sixth-degree fields grow quite quickly.) This relationship is reflected in figure 5.

4 Values of $\zeta_K(\frac{1}{2})$ For Various Number Fields

A natural follow-up to conjecture 3.8 is: What is the value of $\zeta_K(\frac{1}{2}+0i)$ for various K? It is an open conjecture whether $\zeta_K(\frac{1}{2}) < 0$ for all number fields K, but there is a preponderance of empirical data suggesting that it is true. [Tol97, 1314] First, for the Riemann zeta function:

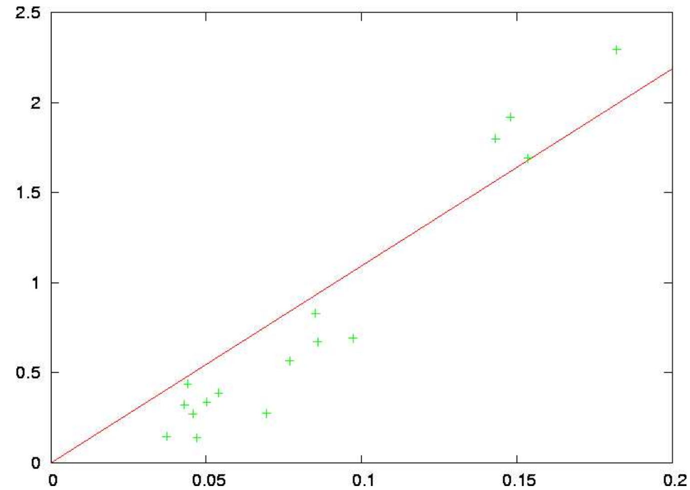

Figure 4: t_0 vs. $\frac{1}{\log|\mathrm{disc}(K)|}$ for cubic fields with signature $(1,1)$ along with the best-fit line which has a slope of 10.942.

Theorem 4.1.

$$\zeta\Big(\frac{1}{2}\Big) < 0.$$

Proof. Consider the meromorphic extension of ζ given in the proof of theorem 2.6: $\zeta(s) = \dfrac{1}{1-2^{1-s}}\zeta_2(s)$. Then $\zeta(\frac{1}{2}) = -\dfrac{1}{\sqrt{2}-1}\zeta_2(1/2)$ so it remains to show that $\zeta_2(1/2) > 0$. This is easily shown:

$$\zeta_2\Big(\frac{1}{2}\Big) = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} - \cdots$$

Grouping pairs of terms:

$$\zeta_2\Big(\frac{1}{2}\Big) = (1 - \frac{1}{\sqrt{2}}) + (\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}) + \cdots,$$

all pairs of terms have a positive sum. $\square$

Corollary 4.2. Assuming the Riemann Hypothesis, ζ is negative on $(0,1)$.

Proof. The Riemann zeta function is real-valued on the real line, and the Riemann Hypothesis implies that ζ is never zero on $(0,1)$, so it may not ever be positive either by the Intermediate Value Theorem. $\square$

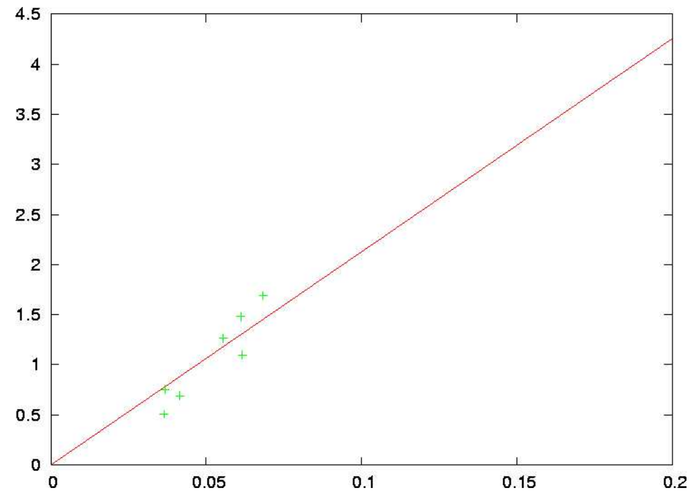

Figure 5: t_0 vs. $\frac{1}{\log|\mathrm{disc}(K)|}$ for sixth-degree fields with signature $(2,2)$. The best-fit line is included; it has a slope of 21.277.

Note that if $\zeta_K(\frac{1}{2}) < 0$ for all number fields K, then the General Riemann Hypothesis implies that all Dedekind zeta functions are negative on $(0,1)$ by similar reasoning.

Applying theorem 2.11, in order to show that $\zeta_K(\frac{1}{2})$ is negative, it remains to show that the product over the non-trivial characters of the L-functions is positive so long as $\mathrm{Aut}(K/\mathbb{Q})$ is abelian. Though the result is not known in general, the problem is most approachable when K is a quadratic extension of $\mathbb{Q}$. In these cases, there is only one non-trivial L-function. For example,

Theorem 4.3. Let $K = \mathbb{Q}[\sqrt{5}]$. Then $\zeta_K(\frac{1}{2}) < 0$.

Proof. As above, it suffices to show that the quadratic L-function of conductor 5 is always positive. Then noting that 1 and 4 are squares mod 5 and 2 and 3 are not (and letting χ be the only non-trivial quadratic character mod 5):

$$L(s,\chi) = \sum_{n=1}^{\infty} \left(\frac{1}{(5n+1)^s} - \frac{1}{(5n+2)^s} - \frac{1}{(5n+3)^s} + \frac{1}{(5n+4)^s} \right).$$

Then evaluating just the summand at $s = \frac{1}{2}$, $L\left(\frac{1}{2},\chi\right)$ is the sum as n varies from

1 to ∞ of

$$\frac{1}{\sqrt{(5n+1)(5n+2)(5n+3)(5n+4)}}\Big(\sqrt{5n+2}\sqrt{5n+3}\sqrt{5n+4} - \sqrt{5n+1}\sqrt{5n+3}\sqrt{5n+4} - \sqrt{5n+1}\sqrt{5n+2}\sqrt{5n+4} + \sqrt{5n+1}\sqrt{5n+2}\sqrt{5n+3}\Big).$$

The result will then follow if it can be shown that the numerator is positive. The numerator is equal to:

$$\begin{aligned}\sqrt{5n+3}\sqrt{5n+4}(\sqrt{5n+2} - \sqrt{5n+1}) + \sqrt{5n+1}\sqrt{5n+2}(\sqrt{5n+3} - \sqrt{5n+4}) \\ \geq \sqrt{5n+1}\sqrt{5n+2}\big((\sqrt{5n+2} - \sqrt{5n+1}) - (\sqrt{5n+4} - \sqrt{5n+3})\big).\end{aligned}$$

The difference $(\sqrt{5n+2} - \sqrt{5n+1}) - (\sqrt{5n+4} - \sqrt{5n+3})$ is the difference in the slopes of two secant lines; these slopes decrease as n increases because $\sqrt{x}$ is concave down. $\square$

4.1 Methods

In order to test the hypothesis that $\zeta_K(\frac{1}{2}) < 0$ for all K, the algebra package PARI was used. Quadratic fields were tested, so it required showing that quadratic L-functions are positive for $s = \frac{1}{2}$. In order to facilitate theoretical work, however, it was not simply tested whether or not $L(\frac{1}{2}, \chi) > 0$; as in the proof of theorem 4.3, sequences of length disc(K) were tested. The purpose of testing strings of this length is that it motivates theoretical proof by a similar method as above.

The L-functions were initialized by creating a vector χ_i of length disc(K) that contained the values of $\chi(i)$ for $1 \leq i \leq \text{disc}(K)$. In particular, χ_i has $\varphi(\text{disc}(K))$ non-zero entries as $\chi(n) = 0$ if n is not an element of $(\mathbb{Z}/\text{disc}(K)\mathbb{Z})^*$. Let $K = \mathbb{Q}[\sqrt{p}]$ with p prime. Then if p is $1 \mod 4$, then the discriminant of K is p, and entries of χ_i were simply given by 1 if i is a square $\mod p$ and -1 if i is not a square $\mod p$. For $p \equiv 3 \mod 4$, the discriminant of K is $4d$, so there are fewer entries in χ_i than simply $4d-1$. The vector was initialized to zero, and then the values of χ_i were then filled in for values of i that were prime and less than $4d$. The remaining elements of $(\mathbb{Z}/\text{disc}(K)\mathbb{Z})^*$, those that are coprime with $4d$ but not prime were then calculated based on their prime factorizations.

Finally, the tests were performed for fields $\mathbb{Q}[\sqrt{p}]$ for primes up to 10^4, and the first 1000 terms were tested.

4.2 Findings

The results were verified in all cases tested. The computations suggest that an algebraic solution may be plausible for the general quadratic case. For higher degree extensions of $\mathbb{Q}$, however, the summation formula is less useful than the product formula as $L(s, \chi)$ is not, in general, real for s real, and the formula may only be directly applied for K with abelian $\mathrm{Aut}(K/\mathbb{Q})$. A simple example requires looking at a cubic character; here two-thirds of the non-zero values are mapped to either $e^{\frac{2\pi i}{3}}$ or $e^{\frac{4\pi i}{3}}$. (It is important to note, however, that the product of all of the cubic L-functions of a given conductor must be real for s real as the product is $\zeta_k(s)$.)

It is also interesting to note that $\zeta_K(\frac{1}{2})$ being negative for all K is a surprising asymmetry for an analytic function satisfying a functional equation relating $\zeta_K(s)$ to $\zeta_K(1-s)$. In particular, the zeroes of the zeta functions in the critical strip are distributed symmetrically about the real axis and the line $\mathrm{Re}(s) = \frac{1}{2}$ despite the fact that $\zeta_K(\frac{1}{2})$ is not zero.

5 Future of the Riemann Hypothesis

One of the most important applications of the Riemann Hypothesis today is in primality testing. The RSA encryption algorithm, in which factoring large numbers that are the product of exactly two primes is equivalent to cracking the cipher, has brought primality testing into the forefront of computational theory. For example, Gary Miller has developed a primality testing algorithm that runs in $O(\log^4 n)$ time assuming the GRH. The search for a fast and unconditional algorithm ultimately led in 2004 to the discovery of a comparatively very fast algorithm for primality testing that requires only $O(\log^{15/2} n)$ time to compute. The so-called AKS algorithm's validity is unconditional; that is, its primality testing capability does not depend on the truth of the RH. This is a fairly astounding result and shows that proving the

Riemann Hypothesis (in one form or another) may be helpful to the study of primes, but there may still be room for improvement without the GRH. [Sar04, 4]

Further, the testing of the Riemann Hypothesis itself does not seem to be yielding to computational force yet; that is, computations have not yet disproved the RH (or GRH) which might lead one to assume that either the scale of computation must be drastically increased to find a counterexample, or the RH is true. Unfortunately, at the present time, computations for large discriminants (10^{10} and larger) are slow and resource-intensive. Because the algorithm implemented in GP/Pari requires calculating many invariants of the field before computing ζ_K, it requires on the order of 1 gigabyte of memory. (Some calculations required upwards of 3 to 4 gigabytes while others required only a few kilobytes.) Further, a long list of primes must be computed to increase the precision of calculations and have a list that can be accessed on-the-fly. Further, calculation times were anywhere from 10 seconds to almost a week.

Locating the low zeros of the Dedekind zeta functions took a couple of minutes for low discriminants (on the order of 100) and a couple of days for large discriminants (on the order of 10^{10}). Computation times were significantly larger for counting zeros on the critical line, ranging from minutes to the better part of a week. All computations were performed on a 3.60 GHz Intel Xeon. These CPUs use a 64-bit architecture and have a 2 megabyte on-chip cache.

Future theoretical work might try to prove the relation in theorem 3.8. Interestingly, the theorem has not even been shown conditionally; it has only been verified computationally for a range of finite discriminants. Any unconditional bound better than the current would be welcomed, as the current bound is $t_0(K)$ is bounded above by a constant plus $o(1)$ which does not relate any properties of K to $t_0(K)$ despite a fair deal of evidence that there is such a relation.

Lastly, an obvious extension of the work done here is to attempt the same computations with larger degree fields. Despite the relatively slow computations of ζ_K for large discriminants (and thus large degree), they are not so insurmountable that the calculations cannot be completed given a reasonable amount of time, nor are they outside of the reach of standard computing technology. An interesting twist, however, is a paper written by Wim van Dam suggesting fast algorithms for approximating the zeros of some zeta functions using quantum computers. [vD04]

A Terms and Notation

$\propto$	$f(x) \propto g(x)$ means that $f(x) = cg(x)$ for some $c \in \mathbb{R}$.
$\|\| \cdot \|\|$	For an element x of a ring A, $\|\|x\|\|$ is the order of A/xA.
Class Number	Two ideals A, B are said to be equivalent if there exist nonzero α, β such that $\alpha A = \beta B$. The class number is the number of classes of ideals under this equivalence relation. [Mar77, 5] The class number of $\mathbb{Z}$ is 1. An important (and computationally difficult) problem in algebraic number theory is finding the class number of the ring of integers for an arbitrary number field.
Dedekind Ring	A ring where every ideal may be factored into prime ideals. Equivalently, Equivalently, Dedekind rings are Noetherian, integrally-closed rings where all nonzero prime ideals are maximal. [Lon77, 1] Examples include the ring of integers of a number field.
Discriminant	An invariant of a number field K (under automorphism) derived from the determinant of the matrix whose entries are $\sigma_i w_j$ where σ_i are the embeddings of K in $\mathbb{C}$ and w_j are n elements of K. [Lon77, 80-81]
Embedding	An embedding of a number field K is a homomorphism $K \to K' \subset \mathbb{C}$ such that the K', the image of K is isomorphic to K. Such homomorphisms must map $\mathbb{Q}$ to itself. Consider the field $\mathbb{Q}[x]/f(x)$ for irreducible $f(x) \in \mathbb{Z}[x]$. Then all real embeddings take roots of f to values of the same degree, and complex embeddings take them to complex value of the same degree. It becomes clear why complex embeddings should come in pairs when noting that they rotate a root r around the circle $\|r\|e^{i\theta}$.
Ideal	A subset of a ring which is closed under addition and scalar multiplication (where scalars are elements of the ring).
Number Field	A number field is a subfield of $\mathbb{C}$ with finite degree over $\mathbb{Q}$. [Mar77, 12]
$o(\cdot)$	$f(x) = o(g(x))$ if and only if $\frac{f(x)}{g(x)} \to 0$ as x tends to some limit; unless otherwise specified, the implied limit is $x \to \infty$. Generally, $o(\cdot)$ is used as an error term meaning that the error term is $o(\cdot)$.
$O(\cdot)$	$f(x) = O(g(x))$ for functions $f, g : \mathbb{R} \to \mathbb{R}$ if and only if there exists $C \geq 0, x_0 \in \mathbb{R}$ such that for all $x \geq x_0$, $f(x) \leq Cg(x)$. Equivalently, $\frac{f(x)}{g(x)}$ is bounded as $x \to \infty$. As with $o(\cdot)$, it is generally used to bound an error term. In this paper, $O(\cdot)$ may also be used to refer to the limit as $\|x\| \to \infty$ which is be made explicit.
Prime Ideal	An ideal P of a ring R such that P/R is an integral domain. In general, in the ring of integers of a number field, one cannot expect to be able to uniquely factor every number. Instead, in Dedekind rings, there is a unique factorization of ideals into prime ideals.
Pole	A function f has a pole at z_0 if $\lim_{z \to z_0} f(z) = \infty$.

Regulator	An invariant of a number field that appears in the class number formula. For details, see [Mar77, chapter 6].
Residue	In complex analysis, the residue of a function f at z_0 is the coefficient of $(z - z_0)^{-1}$ in the Laurent series of f. In particular, f has a simple pole as z_0, its residue is $\lim_{z \to z_0}(z - z_0)f(z)$.
Signature	The ordered pair (r_1, r_2) where r_1 is the number of real embeddings of a number field K and r_2 is the number of conjugate pairs of complex embeddings. Note that $r_1 + 2r_2$ is the degree of K.
Simple Pole	A pole z_0 of a function f where $\lim_{z \to z_0}(z - z_0)f(z)$ exists (and is equal to the residue.)
Ring of Integers	The ring of integers of a number field K is the set of all roots in K of monic polynomials in $\mathbb{Z}[x]$.

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